ENERGY PROPERNESS AND SASAKIAN-EINSTEIN METRICS

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Abstract. In this paper, we show that the existence of Sasakian-Einstein metrics is closely related to the properness of corresponding energy functionals. Under the condition that admitting no nontrivial Hamiltonian holomorphic vector field, we prove that the existence of Sasakian-Einstein metric implies a Moser-Trudinger type inequality. At the end of this paper, we also obtain a Miyaoka-Yau type inequality in Sasakian geometry.

1. Introduction

An odd dimensional Riemannian manifold (M, g) is said to be a Sasakian manifold if the cone manifold $(C(M), \tilde{g}) = (M \times R^+, \quad r^2 g + dr^2)$ is Kähler. In this paper, we suppose that $dimM = 2m + 1$. Furthermore, Sasakian manifold (M, g) is said to be Sasakian-Einstein if the Ricci tensor of g satisfies the Einstein condition. It is well known that the Kähler cone $(C(M), \tilde{g})$ must be a Calabi-Yau cone if (M, g) is a Sasakian-Einstein manifold. Recently, Sasakian-Einstein metrics have attract increasing attention, as they provide rich source of constructing new Einstein manifolds in odd dimensions and its important role in the superstring theory, see references [5, 6, 7, 8, 9, 10, 11, 16, 17, 18, 19, 12, 15, 21, 22, 23, 24, 36].

Sasakian manifolds can be studied from many view points as they have many structures. A Sasakian manifold (M, g) has a contact structure (ξ, η, Φ), and it also has a one dimensional foliation $\mathcal{F}_\xi$, called the Reeb foliation. Here, the Killing vector field ξ is called the characteristic or Reeb vector field, η is called the contact 1-form, Φ is a $(1, 1)$ tensor field which defines a complex structure on the contact sub-bundle $\mathcal{D} = ker\eta$. In the following, a Sasakian manifold will be denoted by (M, ξ, η, Φ, g), the quadruple (ξ, η, Φ, g) will be called by a Sasakian structure on manifold M. In a natural way, a Sasakian structure (ξ, η, Φ, g) induce a transverse holomorphic structure and a transverse Kähler metric on the foliation $\mathcal{F}_\xi$. In this paper, we may change Sasakian structure, but always fix the Reeb vector field ξ and the transverse holomorphic structure on $\mathcal{F}_\xi$.

Fixed a transverse holomorphic structure on $\mathcal{F}_\xi$, we have a splitting of the complexification of the bundle $\wedge^1_B(M)$ of basic one forms on M, $\wedge^1_B(M) \otimes C = \wedge^{1,0}_B(M) \oplus \wedge^{0,1}_B(M)$, and then we have the decomposition of d, i.e. $d = \partial_B + \bar{\partial}_B$. We have the basic cohomology groups $H^{i,j}_B(M, \mathcal{F}_\xi)$ which enjoy many of the same properties as the Dolbeault cohomology of a Kähler structure. We also have the transverse Chern-Weil theory and can define the basic Chern classes $c^B_k(M, \mathcal{F}_\xi)$. For the detail, see [8]. Given a transverse Kähler structure g^T, one can define the transverse Levi-Civita connection ∇^T on the normal bundle $\nu(\mathcal{F}_\xi) = TM/L\xi$, and then one can define the transverse Ricci curvature Ric^T, see section 2 for details. We denote the related Ricci form by ρ^T, it is easy to see that ρ^T is a

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closed basic $(1,1)$-form and the basic cohomology class $\frac{1}{2\pi}[\rho^T]_B = c_1^B(M, \mathcal{F}_\xi)$ is the basic first Chern class. A Sasakian metric (ξ, η, Φ, g) is said to be transversely Kähler-Einstein if it's transverse Ricci form satisfies $\rho^T = \mu d\eta$. It's easy to see that a Sasakian metric (ξ, η, Φ, g) is Sasakian-Einstein then it must be transversely Kähler-Einstein and $\rho^T = (m+1)d\eta$. So, a necessary condition for the existence of Sasakian-Einstein metric on M is that there exists a Sasakian structure (ξ, η, Φ, g) such that $2\pi c_1^B(M, \mathcal{F}_\xi) = (m+1)[d\eta]_B$.

Given a Sasakian structure (ξ, η, Φ, g) on M, Let's denote the space of all smooth basic real function φ (i.e. $\xi\varphi \equiv 0$) on (M, ξ, η, Φ, g) by $C_B^\infty(M, \xi)$. Set

$$\mathcal{H}(\xi, \eta, \Phi, g) = \{\varphi \in C_B^\infty(M, \xi) : \eta_\varphi \wedge (d\eta_\varphi)^n \neq 0\}, \tag{1.1}$$

where

$$\eta_\varphi = \eta + \sqrt{-1}\frac{1}{2}(\bar{\partial}_B - \partial_B)\varphi, \quad d\eta_\varphi = d\eta + \sqrt{-1}\partial_B\bar{\partial}_B\varphi. \tag{1.2}$$

For any $\varphi \in \mathcal{H}$, $(\xi, \eta_\varphi, \Phi_\varphi, g_\varphi)$ is also a Sasakian structure on M, where

$$\Phi_\varphi = \Phi - \xi \otimes (d_B^c\varphi) \circ \Phi, \quad g_\varphi = \frac{1}{2}d\eta_\varphi \circ (Id \otimes \Phi_\varphi) + \eta_\varphi \otimes \eta_\varphi. \tag{1.3}$$

Furthermore, $(\xi, \eta_\varphi, \Phi_\varphi, g_\varphi)$ and (ξ, η, Φ, g) have the same transversely holomorphic structure on $\nu(\mathcal{F}_\xi)$ and the same holomorphic structure on the cone $C(M)$ (Proposition 4.2 in [15], also [8]). Obviously, those deformations of Sasakian structure deform the transverse Kähler form in the same basic $(1,1)$ class. We call this class the basic Kähler class of the Sasakian manifold (M, ξ, η, Φ, g).

As in the Kähler case, one can define Aubin's functionals $I_{d\eta}$, $J_{d\eta}$, Ding and Tian's energy functional $F_{d\eta}$, and Mabuchi's K-energy functional $\mathcal{V}_{d\eta}$ on the space $\mathcal{H}(\xi, \eta, \Phi, g)$, see section 3 for details. We say the energy functional $F_{d\eta}$ (or $\mathcal{V}_{d\eta}$) is proper if $\limsup_{i\to+\infty} F_{d\eta}(\varphi_i) = +\infty$ whenever $\lim_{i\to+\infty} J_{d\eta}(\varphi_i) = +\infty$, where $\varphi_i \in \mathcal{H}(\xi, \eta, \Phi, g)$. We say two Sasakian structures are compatible with each other if they have the same Reeb vector field and the same transverse holomorphic structure.

In Kähler geometry, Tian [33] have show that the existence of Kähler-Einstein metric is equivalent with the properness of F energy functional on a compact Kähler manifold with positive Chern class and without any nontrivial holomorphic fields. In this paper, under the condition that without nontrivial Hamiltonian holomorphic vector field (see section 2 for details), we generalize Tian's result in [33] to the Sasakian case. In fact, we obtain the following theorem.

Main Theorem *Let (M, ξ, η, Φ, g) be a compact Sasakian manifold with $[d\eta]_B = \frac{2\pi}{m+1}c_1^B(M, \mathcal{F}_\xi)$ and without any nontrivial Hamiltonian holomorphic vector field. Then M has a Sasakian-Einstein structure compatible with (ξ, η, Φ, g) if and only if the functional $F_{d\eta}$ (or the K-energy functional $\mathcal{V}_{d\eta}$) is proper in the space $\mathcal{H}(\xi, \eta, \Phi, g)$.*

We will follow Tian's method, and the discussion in [26] by Phong, Song, Strum and Weinkove. It's easy to see that if a Sasakian structure (ξ', η', Φ', g') is compatible with (ξ, η, Φ, g), then we have $[d\eta']_B = [d\eta]_B \in H_B^{1,1}(\mathcal{F}_\xi)$, by transverse $\partial\bar{\partial}$ lemma ([14]), there exists a basic function $\varphi \in \mathcal{H}(\xi, \eta, \Phi, g)$ such that

$$d\eta' = d\eta + dd_B^c\varphi, \quad \textit{and} \quad \eta' = \eta + d_B^c\varphi + \zeta \tag{1.4}$$

where ζ is closed basic one form, $d_B^c = \frac{\sqrt{-1}}{2}(\bar{\partial}_B - \partial_B)$. So, the existence problem of Sasakian-Einstein metric compatible with (ξ, η, Φ, g) can be reduced to solving

the following transverse Monge-Ampère equation,

$$\frac{(d\eta+\sqrt{-1}\partial_B\overline{\partial}_B\varphi)^m\wedge\eta}{(d\eta)^m\wedge\eta}=\exp(h_{d\eta}-(m+1)\varphi), \tag{1.5}$$

where $\varphi\in\mathcal{H}(\xi,\eta,\Phi,g)$ and $h_{d\eta}$ is a smooth basic function which satisfies $\rho^T=(m+1)d\eta+\sqrt{-1}\partial_B\bar{\partial}_B h_{d\eta}$ and $\int_M\exp(h_{d\eta})(d\eta)^m\wedge\eta=\int_M(d\eta)^m\wedge\eta=V$. In order to use the continuity method, we consider the following family of equations

$$\frac{(d\eta+\sqrt{-1}\partial_B\overline{\partial}_B\varphi)^m\wedge\eta}{(d\eta)^m\wedge\eta}=\exp(h_{d\eta}-t(m+1)\varphi), \tag{1.6}$$

where $t\in[0,1]$. By El-Kacimi's ([14]) generalization of Yau's estimate ([35]) for transverse Monge-Ampère equations, to solve the transverse Monge-Ampère equation (1.5), it is sufficient to obtain a priori uniform C^0 estimate of solution φ of the transverse Monge-Ampère equation (1.6). Different than the Kähler case, the transverse Monge-Ampère equation (1.6) only give bounds on the transverse Ricci curvature which does not lead a lower bound of the Ricci curvature by a positive constant. So, we can not apply the Myers theorem directly to obtain an estimate on the diameter and the lower bound of the Green's function. It's pointed out by Sekiya ([28]), through $\mathcal{D}$-homothetic deformation of Sasakian structure, one can get the desired estimates in the Sasakian case. In section 4 (theorem 4.3), we show that the properness of energy functional $F_{d\eta}$ (or the K-energy functional $\mathcal{V}_{d\eta}$) implies the C^0 estimates, and so we get a existence result for Sasakian-Einstein metric.

To prove the existence of Sasakian-Einstein metric implies the properness of energy functional, we use the backward continuity method. In order to apply the implicity function theorem at time $t=1$, we should consider the eigenspace corresponding to the eigenvalue $-4(m+1)$ of the basic Laplacian of the Sasakian-Einstein metric. Thanks to A. Futaki, H. Ono and G. Wang's result [15], we know that the above eigenspace is not empty if and only if there exists a nontrivial Hamiltonian holomorphic vector field. In section 6 (theorem 6.1), under the condition that without nontrivial Hamiltonian holomorphic vector field, we obtain the following Moser-Trudinger inequality on Sasakian-Einstein manifold $(M,\xi,\eta_{SE},\Phi_{SE},g_{SE})$, i.e. there exist uniform positive constants C_1, C_2, such that

$$F_{d\eta_{SE}}(\varphi)\geq C_1J_{d\eta_{SE}}(\varphi)-C_2, \tag{1.7}$$

for all $\varphi\in\mathcal{H}(\xi,\eta_{SE},\Phi_{SE},g_{SE})$. In view of the cocycle identity of $F_{d\eta}$ and properties of $J_{d\eta}$ (see section 3, lemma 3.1 and lemma 3.2), the inequality holds for every Sasakian structure (ξ,η,Φ,g) which compatible with the Sasakian-Einstein structure $(\xi,\eta_{SE},\Phi_{SE},g_{SE})$. On the other hand, the relation (3.25) implies that the Moser-Trudinger type inequality (1.7) also be valid for the $\mathcal{K}$-energy $\mathcal{V}_{d\eta}$.

Given a Sasakian structure (ξ,η,Φ,g) on M with $2\pi c_1^B(M)=(m+1)[d\eta]_B$, we denote $\mathcal{S}(\xi,\bar{J})$ to be the set of all Sasakian structures which compatible with (ξ,η,Φ,g). By definition 2.6 and proposition 2.7, it's easy to see that the integral

$$\int_M(2c_2^B(M,\mathcal{F}_\xi)-\frac{m}{m+1}c_1^B(M,\mathcal{F}_\xi)^2)\wedge\frac{(\frac{1}{2}d\eta')^{m-2}}{(m-2)!}\wedge\eta' \tag{1.8}$$

is independent of the choice of a Sasakian structure $(\xi,\eta',\Phi',g')\in\mathcal{S}(\xi,\bar{J})$. By direct calculation, see lemma 7.2, if there exists a Sasakian-Einstein structure (or equivalently a Sasakian structure with constant scalar curvature)in $\mathcal{S}(\xi,\bar{J})$, then the integral (1.8) must be nonnegative, this inequality will be called a Miyaoka-Yau

type inequality. In section 7, we discuss this Miyaoka-Yau type inequality under a more weak condition, and show that the energy function $F_{d\eta}$ (or Mabuchi's $\mathcal{K}$-energy $\mathcal{V}_{d\eta}$) bounded below will implies this weak condition, see theorem 7.3 and proposition 7.5 for details.

This paper is organized as follows. In Section 2, we will recall some preliminary results about Sasakian geometry. In section 3, we introduce energy functionals in Sasakian geometry. In section 4, we consider the transverse Monge-Ampère equation, and give a existence result of Sasakian-Einstein metric. In section 5, we use the Sasakian-Ricci flow to get a smoothing lemma. In section 6, we give a proof of the Moser-Trudinger type inequality (1.7), and finish the proof of the main theorem. In the last section, we obtain a Miyaoka-Yau type inequality in Sasakian geometry.

2. Preliminary Results in Sasakian geometry

2.1. **Transverse Kähler structure.** Let (M, ξ, η, Φ, g) be a $2m+1$-dimensional Sasakian manifold, and let $\mathcal{F}_\xi$ be the characteristic foliation generated by ξ. Firstly, Let us recall that a Sasakian structure induce a transverse Kähler structure on the foliation $\mathcal{F}_\xi$. A transverse holomorphic structure on $\mathcal{F}_\xi$ is given by an open covering $\{U_i\}_{i\in A}$ of M and local submersion $f_i : U_i \to C^n$ with fibers of dimension 1, (the leaves of the foliation $\mathcal{F}_\xi|_{U_j}$ on U_j coincide with the fibers of the map f_i, leave is the image of the flow of ξ), such that for $i, j \in A$ there is a holomorphic isomorphism θ_{ij} of open sets of C^n such that $f_i = \theta_{ij} \circ f_j$ on $U_i \cap U_j$.

In order to consider the deformations of Sasakian structures, we consider the quotient bundle of the foliation $\mathcal{F}_\xi$, $\nu(\mathcal{F}_\xi) = TM/L\xi$. The metric g gives a bundle isomorphism σ between $\nu(\mathcal{F}_\xi)$ and the contact sub-bundle $\mathcal{D} = Ker\eta$, where $\sigma : \nu(\mathcal{F}_\xi) \to \mathcal{D}$ defined by

$$\sigma([X]) = X - \eta(X)\xi.$$

By this isomorphism, $\Phi|_{\mathcal{D}}$ induces a complex structure $\bar{J}$ on $\nu(\mathcal{F}_\xi)$. Since the Nijenhuis torsion tensor of Φ satisfies

$$N_\Phi(X, Y) = -d\eta(X, Y) \otimes \xi.$$

So, $(\nu(\mathcal{F}_\xi), \bar{J}) \cong (\mathcal{D}, \Phi|_{\mathcal{D}})$ gives $\mathcal{F}_\xi$ a transverse holomorphic structure. Then $(\mathcal{D}, \Phi|_{\mathcal{D}}, d\eta)$ gives $\mathcal{F}_\xi$ a a transverse Kähler structure with transverse Kähler form $\frac{1}{2}d\eta$ and metric g^T defined by $g^T(\cdot,\cdot) = \frac{1}{2}d\eta(\cdot, \Phi\cdot)$.

In the following we say that a Sasakian structure (ξ, η', Φ', g') have the same transverse holomorphic transverse structure with that of (ξ, η, Φ, g), means that it satisfies

$$\bar{J} \circ \pi_\nu = \pi_\nu \circ \Phi' \tag{2.1}$$

where π_ν is the projection $\pi_\nu : TM \to \nu(\mathcal{F}_\xi)$.

Definition 2.1. *We define $\mathcal{S}(\xi, \bar{J})$ to be the set of all Sasakian structures which have the same Reeb vector field and the same transverse holomorphic structure with (ξ, η, Φ, g), i.e. all Sasakian structures which compatible with (ξ, η, Φ, g).*

Definition 2.2. *Fixed a transverse holomorphic structure $(\nu(\mathcal{F}_\xi), \bar{J})$ on the characteristic foliation $\mathcal{F}_\xi$. A complex vector field X on M is called a transverse holomorphic vector field if it satisfies:*

(1) $\pi[\xi, X] = 0$;
(2) $\bar{J}(\pi(X)) = \sqrt{-1}\pi(X)$;
(3) $\pi([Y, X]) - \sqrt{-1}\bar{J}\pi([Y, X]) = 0$, $\forall Y$ *satisfying* $\bar{J}\pi(Y) = -\sqrt{-1}\pi(Y)$,

where π *is the projection* $\pi : TM \to \nu(\mathcal{F}_\xi)$. *Given a transverse Kähler form* $d\eta$. *Let* ψ *be a complex valued basic function, then there is a unique vector field* $V_{d\eta}(\psi) \in \Gamma(T^c M)$ *satisfies: (1)*$\bar{J}(\pi(V_{d\eta}(\psi))) = \sqrt{-1}\pi(V_{d\eta}(\psi))$; *(2)* $\psi = \sqrt{-1}\eta(V_\eta(\psi))$; *(3)* $\bar{\partial}_B\psi = -\frac{\sqrt{-1}}{2} d\eta(V_\eta(\psi), \cdot)$. *The vector field* $V_\eta(\psi)$ *is called the Hamiltonian vector field of* ψ *corresponding to the transverse Kähler form* $d\eta$. *A complex vector field* X *on* M *is called a Hamiltonian holomorphic vector field if it is transverse holomorphic and is the Hamiltonian vector field of some complex valued basic function* ψ *corresponding to some transverse Kähler form* $d\eta$.

Remark 2.3. *By the definition,* $c\xi$ *is a Hamiltonian holomorphic vector field for any constant* c. *In this paper, without nontrivial Hamiltonian holomorphic vector field means that any Hamiltonian holomorphic vector field must be* 0 *or* $c\xi$.

Given a Sasakian structure (ξ, η, Φ, g), we might identity $\mathcal{D}$ with $\nu(\mathcal{F}_\xi)$ by the isomorphism. However, it's better to distinguish them, since under the deformations of Sasakian structure, the contact sub-bundle $\mathcal{D}$ changes, while $\nu(\mathcal{F}_\xi)$ keeps fixed. But for simplicity of notation, we will use the same notation if there is no confusion, especially if we do not consider deformations. From the transverse Kähler structure $(\mathcal{D}, \Phi|_{\mathcal{D}}, d\eta)$, one can define the transverse Levi-Civita connection ∇^T on $\mathcal{D}$ by

$$\nabla^T_X Y = \begin{cases} (\nabla_X Y)^p, & X \in \mathcal{D}, \\ [\xi, Y]^p, & X = \xi, \end{cases} \tag{2.2}$$

where ∇ is the Levi-Civita connection with respect to the Riemannian metric g, Y is a section of $\mathcal{D}$ and X^p the projection of X onto $\mathcal{D}$. It is easy to check that the transverse Levi-Civita connection is torsion-free and metric compatible. The transverse curvature tensor and transverse Ricci curvature are defined by

$$R^T(V, W)Z = \nabla^T_V \nabla^T_W Z - \nabla^T_W \nabla^T_V Z - \nabla^T_{[V,W]} Z, \tag{2.3}$$

$$Ric^T(X, Y) = < R^T(X, e_i)e_i, Y >_g, \tag{2.4}$$

where e_i is an orthonormal basis of $\mathcal{D}$, $X, Y, Z \in \mathcal{D}$ and $V, W \in TM$. We also have the following relations between the transverse curvature tensor and the Riemann curvature tensor (see [10])

$$\begin{aligned} &R^T(X, Y)Z = R(X, Y)Z - < R(X, Y)Z, \xi > \xi \\ &- < \nabla_Y Z, \xi > \Phi(X) + < \nabla_X Z, \xi > \Phi(Y) + < [X, Y], \xi > \Phi(Z), \end{aligned} \tag{2.5}$$

and

$$Ric^T(X, Y) = Ric(X, Y) + 2g^T(X, Y), \tag{2.6}$$

for $X, Y, Z \in \mathcal{D}$. The transverse Ricci form defined as following

$$\rho^T(X, Y) = Ric^T(\Phi X, Y). \tag{2.7}$$

Definition 2.4. *A Sasakian manifold* (M, ξ, η, Φ, g) *is said to be transversely Kähler-Einstein if*

$$Ric^T = \mu g^T, \quad or \quad \rho^T = \mu(\frac{1}{2} d\eta)$$

for some constant c.

A Sasakian manifold (M,ξ,η,Φ,g) is called a Sasakian-Einstein manifold if g is a Riemannian Einstein metric, i.e. $Ric_g = cg$ for some constant c. It is easy to see that a Sasakian-Einstein must be transversely Kähler-Einstein, the constant c must be $2m$, and

$$Ric^T = (2m+2)g^T, \quad or \quad \rho^T = (m+1)d\eta. \tag{2.8}$$

2.2. **Basic cohomology.** A p-form θ on Sasakian manifold (M,ξ,η,Φ,g) is called basic if

$$i_\xi\theta = 0, \quad L_\xi\theta = 0, \tag{2.9}$$

where i_ξ is the contraction with the Killing vector field ξ, L_ξ is the Lie derivative with respect to ξ. Basic cohomology was introduced by Reinhart in [27]. We begin with a brief review following [34]. It is easy to see that the exterior differential preserves basic forms. Namely, if θ is a basic form, so is $d\theta$. Let $\wedge_B^p(M)$ be the sheaf of germs of basic p-forms and $\Omega_B^p(M) = \Gamma(M, \wedge_B^p(M))$ the set of all section of $\wedge_B^p(M)$. The basic cohomology can be defined in a usual way (see [14]). Let $\mathcal{D}^C$ be the complexification of the sub-bundle $\mathcal{D}$, and decompose it into its eigenspaces with respect to $\Phi|_D$, that is

$$\mathcal{D}^C = \mathcal{D}^{1,0} \oplus \mathcal{D}^{0,1}. \tag{2.10}$$

Similarly, we have a splitting of the complexification of the bundle $\wedge_B^1(M)$ of basic one forms on M,

$$\wedge_B^1(M) \otimes C = \wedge_B^{1,0}(M) \oplus \wedge_B^{0,1}(M). \tag{2.11}$$

Let $\wedge_B^{i,j}(M)$ denote the bundle of basic forms of type (i,j). Accordingly, we have the following decomposition

$$\wedge_B^p(M) \otimes C = \oplus_{i+j=p} \wedge_B^{i,j}(M). \tag{2.12}$$

Define ∂_B and $\overline{\partial}_B$ by

$$\begin{array}{rcl} \partial_B & : & \wedge_B^{i,j}(M) \to \wedge_B^{i+1,j}(M); \\ \overline{\partial}_B & : & \wedge_B^{i,j}(M) \to \wedge_B^{i,j+1}(M); \end{array} \tag{2.13}$$

which is the decomposition of d. Let $d_B^c = \frac{1}{2}\sqrt{-1}(\overline{\partial}_B - \partial_B)$ and $d_B = d|_{\wedge_B^p}$. We have $d_B = \overline{\partial}_B + \partial_B$, $d_B d_B^c = \sqrt{-1}\partial_B\overline{\partial}_B$, $d_B^2 = (d_B^c)^2 = 0$. The basic cohomology groups $H_B^{i,j}(M,\mathcal{F}_\xi)$ are fundamental invariants of a Sasakian structure which enjoy many of the same properties as the Dolbeault cohomology of a Kähler structure [8]. On Sasakian manifolds, the $\partial\overline{\partial}$-lemma holds for basic forms.

Proposition 2.5. ([14]) Let θ and θ' be two real closed basic form of type $(1,1)$ on a compact Sasakian manifold (M,ξ,η,Φ,g). If $[\theta]_B = [\theta']_B \in H_B^{1,1}(M,\mathcal{F}_\xi)$, then there is a real basic function φ such that

$$\theta = \theta' + \sqrt{-1}\partial_B\overline{\partial}_B\varphi.$$

Consider the complex bundle $(\mathcal{D}, \Phi|_{\mathcal{D}})$ (or $(\nu(\mathcal{F}_\xi), \bar{J})$) on a Sasakian manifold (M,ξ,η,Φ,g). Let Rm^T be the transverse curvature with respect to the transverse Levi-Civita connection ∇^T. If we choose a local foliate transverse frame $(X_1,\cdots,X_m)$ on the bundle $\mathcal{D}$, then Rm^T can be seen as a matrix valued 2-form

(i.e. $End(\mathcal{D})$-valued). Rm^T is a basic $(1,1)$-form. Let's define the basic (k,k)-form γ_k by the formula

$$det(Id_m + \frac{\sqrt{-1}}{2\pi} Rm^T) = 1 + \sum_{k=1}^{m} \gamma_k. \tag{2.14}$$

Definition 2.6. γ_k is a closed basic (k,k)-form it represents an element in $H_B^{k,k}(M, \mathcal{F}_\xi)$ that is called the basic k^{th} Chern class and denoted by $c_k^B(M, \mathcal{F}_\xi)$.

We have the following proposition.

Proposition 2.7. **([8], proposition 7.5.21)** *The basic Chern classes $c_k^B(M, \mathcal{F}_\xi)$ are independent of the choice of a Sasakian structure in $\mathcal{S}(\xi, \bar{J})$*

Let $\rho^T = Ric^T(\Phi\cdot, \cdot)$ be the transverse Ricci form of the Sasakian structure (ξ, η, Φ, g). ρ^T is a real closed basic $(1,1)$-form and the basic cohomology class $\frac{1}{2\pi}[\rho^T]_B = c_1^B(M, \mathcal{F}_\xi)$ is the basic first Chern class. We say that the basic first Chern class of $(M, \mathcal{F}_\xi)$ is positive (negative , null resp.) if it contains a positive (negative, null resp.) representation. By the definition and (2.8), a necessary condition for the existence of Sasakian-Einstein metric (M, ξ, η, Φ, g) is $2\pi c_1^B(M, \mathcal{F}_\xi) = (m+1)[d\eta]_B$.

On Sasakian manifold (M, ξ, η, Φ, g), the basic Laplacian is defined by

$$\triangle_B \psi = \frac{4m\sqrt{-1}\partial_B \bar{\partial}_B \psi \wedge (d\eta)^{m-1} \wedge \eta}{(d\eta)^m \wedge \eta}, \tag{2.15}$$

for any basic function ψ. It is well known that the basic Laplacian is equal to the restriction of the Riemannian Laplacian $\triangle_g$ on basic functions, i.e $\triangle_B \psi = \triangle_g \psi$ for any basic function ψ.

2.3. **Transformations of Sasakian structures.** Let (ξ, η, Φ, g) be a Sasakian structure on M, for every real basic function $\varphi \in \mathcal{H}(\xi, \eta, \Phi, g)$, we can obtain a new Sasakian structure $(\xi, \eta_\varphi, \Phi_\varphi, g_\varphi)$. Here $\eta_\varphi, \Phi_\varphi, g_\varphi$ is defined as that in (1.2) and (1.3). We know that the above deformations fix the Reeb vector field ξ and the transverse holomorphic structure.

Let (ξ', η', Φ', g') be an another Sasakian structure which is compatible with (ξ, η, Φ, g), then there exists a basic function $\varphi \in \mathcal{H}(\xi, \eta, \Phi, g)$ such that $d\eta' = d\eta + dd_B^c \varphi$ and $\eta' = \eta + d_B^c \varphi + \zeta$, where ζ is closed basic one form. Since the difference between η' and η is a basic one form, it is easy to see that

$$(d\eta')^m \wedge \eta' = (d\eta')^m \wedge \eta, \tag{2.16}$$

and

$$\int_M (d\eta')^m \wedge \eta' = \int_M (d\eta)^m \wedge \eta = V. \tag{2.17}$$

In the following, $\rho_{d\eta}^T$ denotes the transverse Ricci form with respect to the transverse Kähler metric $d\eta$. In local foliation coordinates $(x, z_1, \cdots, z_m)$, we have

$$\rho_{d\eta}^T = -\sqrt{-1}\partial_B \overline{\partial}_B \log det((d\eta)_{i\bar{j}}). \tag{2.18}$$

Globally, the difference between two transverse Ricci form can be expressed by

$$\rho_{d\eta}^T - \rho_{d\eta'}^T = \sqrt{-1}\partial_B \overline{\partial}_B \log(\frac{(d\eta')^m \wedge \eta'}{(d\eta)^m \wedge \eta}). \tag{2.19}$$

From the above formula, we know that to find a Sasakian-Einstein structure compatible with (ξ, η, Φ, g) is equivalent to solve the transverse Monge-Ampère equation (1.5).

Let's recall one class special deformations of Sasakian structure,

$$\xi_s = s^{-1}\xi, \quad \eta_s = s\eta, \quad \Phi_s = \Phi, \quad g_s = sg + s(s-1)\eta \otimes \eta, \tag{2.20}$$

where s is constant. These were called $\mathcal{D}$-homothetic deformations by Tanno [30]. These deformations do not deform the characteristic foliation and the contact bundle $\mathcal{D}$, but only rescale the Reeb field ξ and contact 1-form η. If (ξ, η, Φ, g) be a transverse Kähler Einstein Sasakian metric with positive transverse Ricci curvature, i.e. $Ric_g^T = \mu g^T$ $(Ric = (\mu - 2)g + (2n + 2 - \mu)\eta \otimes \eta)$for positive constant $\mu \in R$. Then, by the $\mathcal{D}$-homothetic transformation $s = \frac{\mu}{2(n+1)}$, we get a Sasakian-Einstein metric $(\xi_s, \eta_s, \Phi_s, g_s)$. Indeed, by the relation formula (2.15) in [30] , we have

$$\begin{array}{rcl} Ric_{g_s} & = & Ric_g - 2(s-1)g + s^{-1}(s-1)(2(2n+1)s + 2ns(s-1))\eta \otimes \eta \\ & = & \{2n + 2 - \mu + s^{-1}(s-1)(2(2n+1)s + 2ns(s-1))\}\eta \otimes \eta \\ & & +(\mu - 2 - 2(s-1))g \\ & = & \frac{\mu - 2s}{s}\{sg + s(s-1)\eta \otimes \eta\} + ((2n+2)s^2 - \mu s)\eta \otimes \eta \\ & = & 2mg_s. \end{array} \tag{2.21}$$

Through Tanno's $\mathcal{D}$-homothetic transformation, one can prove the following lemma. The proof can be found in [25] and [28], see also proposition 2.6 in [37].

Proposition 2.7. *Let (M, ξ, η, g) be a $2m + 1$-dimensional compact Sasakian manifold with $Ric^T \geq \epsilon g^T$. Suppose that $\phi \in C_B^\infty(M)$ satisfies $\triangle_g \phi \leq \delta$, then we have*

$$-\inf_M \phi \leq \frac{1}{V}\int_M (-\phi)(d\eta)^m \wedge \eta + \frac{C(m)\delta}{\epsilon}, \tag{2.22}$$

where $V = \int_M (d\eta)^m \wedge \eta$ and constant $C(m)$ depends only on m.

From the above proposition, it is easy to conclude the following corollary.

Corollary 2.8. *Let (M, ξ, η, g) be a $2m + 1$-dimensional compact Sasakian manifold and $\varphi \in \mathcal{H}(\xi, \eta, \Phi, g)$ be a potential function with $Ric^T(d\eta_\varphi) \geq \epsilon g_\varphi^T$. Then we have*

$$Osc(\varphi) \leq I_{d\eta}(\varphi) + \frac{\tilde{C}(m)}{\epsilon} + \tilde{C}(M, g), \tag{2.23}$$

where constant $\tilde{C}(M, g)$ depends only on (M, g) and constant $\tilde{C}(m)$ depends only on m. To the definition of $I_{d\eta}$, see section 3.

Proof. Using the fact $\triangle_{d\eta}\varphi_t \geq -4m$ and the Green's formula, we have

$$\sup_M \varphi \leq \frac{1}{V}\int_M \varphi(d\eta)^m \wedge \eta + \tilde{C}(M, g). \tag{2.24}$$

Then, by $\triangle_{d\eta_\varphi}\varphi \leq 4m$ and the above proposition, we have

$$\begin{array}{rcl} Osc(\varphi) & = & \sup_M \varphi_t - \inf_M \varphi_t \\ & \leq & I_{d\eta}(\varphi) + \frac{\tilde{C}(m)}{\epsilon} + \tilde{C}(M, g). \end{array} \tag{2.25}$$

□

3. Energy functionals

Let (ξ,η,Φ,g) be a Sasakian structure on M. We consider following functionals on $\mathcal{H}(\xi,\eta,\Phi,g)$ which are analogous to the ones in Kähler geometry.

$$\begin{array}{lll} I_{d\eta}(\varphi) & := & \frac{1}{V}\int_M \varphi\{(d\eta)^m\wedge\eta-(d\eta_\varphi)^m\wedge\eta\} \\ J_{d\eta}(\varphi) & := & \int_0^1 \frac{1}{s} I_{d\eta}(s\varphi)ds, \\ F^0_{d\eta}(\varphi) & := & J_{d\eta}(\varphi)-\frac{1}{V}\int_M\varphi(d\eta)^m\wedge\eta \\ F_{d\eta}(\varphi) & := & F^0_{d\eta}(\varphi)-\frac{1}{m+1}\log\{\frac{1}{V}\int_M e^{h_{d\eta}-(m+1)\varphi}(d\eta)^m\wedge\eta\} \end{array} \tag{3.1}$$

where $V=\int_M(d\eta)^m\wedge\eta$, and $h_{d\eta}$ is a smooth basic function which satisfies $\rho^T=(m+1)d\eta+\sqrt{-1}\partial_B\bar\partial_B h_{d\eta}$ and $\int_M\exp(h_{d\eta})(d\eta)^m\wedge\eta=V$. Noting that $g(\Phi\cdot,\cdot)=\frac{1}{2}d\eta$, when $\varphi\in C^\infty_B(M)$ we have

$$m\sqrt{-1}\partial_B\bar\partial_B\varphi\wedge(d\eta)^{m-1}\wedge\eta=\frac{1}{4}\triangle\varphi(d\eta)^m\wedge\eta, \tag{3.2}$$

where $\triangle$ is the Laplace of the metric g. Let φ_s be a smooth curve in $\mathcal{H}$, by direct calculation, we have

$$\begin{array}{lll} \frac{d}{ds}I_{d\eta}(\varphi_s) & = & \frac{1}{V}\int_M\dot\varphi_s\{(d\eta)^m-(d\eta_{\varphi_s})^m\}\wedge\eta \\ & & -\frac{1}{4V}\int_M\varphi_s\triangle_{\varphi_s}\dot\varphi_s(d\eta_{\varphi_s})^m\wedge\eta, \end{array} \tag{3.3}$$

$$\frac{d}{ds}J_{d\eta}(\varphi_s)=\frac{1}{V}\int_M\dot\varphi_s\{(d\eta)^m-(d\eta_{\varphi_s})^m\}\wedge\eta, \tag{3.4}$$

$$\frac{d}{ds}F^0_{d\eta}(\varphi_s)=-\frac{1}{V}\int_M\dot\varphi_s(d\eta_{\varphi_s})^m\wedge\eta, \tag{3.5}$$

and

$$\begin{array}{l} \frac{d}{ds}F_{d\eta}(\varphi_s)=-\frac{1}{V}\int_M\dot\varphi_s(d\eta_{\varphi_s})^m\wedge\eta \\ +(\int_M e^{h_{d\eta}-(m+1)\varphi}(d\eta)^m\wedge\eta)^{-1}\int_M\dot\varphi_s e^{h_{d\eta}-(m+1)\varphi}(d\eta)^m\wedge\eta, \end{array} \tag{3.6}$$

where $\dot\varphi_s=\frac{d}{ds}\varphi_s$ and $\triangle_{\varphi_s}$ is the Laplace corresponding with the metric g_{φ_s}. From (3.6), it is easy to that the critical points of $F_{d\eta}$ are transverse Kähler-Einstein metrics.

The following properties can be proved by a similar method as in the Kähler case (see [4], [32], [25], [28]).

Proposition 3.1. *Let C be a constant, then*

$$I_{d\eta}(\varphi+C)=I_{d\eta}(\varphi),\quad J_{d\eta}(\varphi+C)=J_{d\eta}(\varphi),\quad F_{d\eta}(\varphi+C)=F_{d\eta}(\varphi). \tag{3.7}$$

$I_{d\eta}$, $I_{d\eta}-J_{d\eta}$, $J_{d\eta}$ are non-negative functionals on $\mathcal{H}(\xi,\eta,\Phi,g)$, and we have

$$I_{d\eta}(\varphi)\le(m+1)\{I_{d\eta}(\varphi)-J_{d\eta}(\varphi)\}\le mI_{d\eta}(\varphi). \tag{3.8}$$

Let φ_t be a family of basic functions in $\mathcal{H}$, then

$$\frac{d}{dt}\{I_{d\eta}(\varphi_t)-J_{d\eta}(\varphi_t)\}=\frac{-1}{4V}\int_M\varphi_t(\triangle_t\frac{d}{dt}\varphi_t)(d\eta_{\varphi_t})^m\wedge\eta, \tag{3.9}$$

where $\triangle_t$ is the Laplacian corresponding to the metric g_{φ_t}. $F_{d\eta}$ satisfies the following cocycle property, i.e.

$$F_{d\eta}(\psi)+F_{d\eta'}(\phi-\psi)=F_{d\eta}(\phi), \tag{3.10}$$

and

$$F_{d\eta}(\psi)=-F_{d\eta'}(-\psi) \tag{3.11}$$

for all $\phi, \psi \in \mathcal{H}(\xi, \eta, \Phi, g)$ and $d\eta' = d\eta + \sqrt{-1}\partial_B\overline{\partial}_B\psi$. We also have the cocycle condition for $F^0_{d\eta}$.

Lemma 3.2. *Let (ξ, η, Φ, g) and (ξ, η', Φ', g') are two Sasakian structures with the same transverse holomorphic structure on M, and assume that $d\eta = d\eta' + \sqrt{-1}\partial_B\bar{\partial}_B\phi$ for some basic function ϕ. Then, we have*

$$|I_{d\eta'}(\varphi + \phi) - I_{d\eta}(\varphi)| \leq (m + 1)Osc(\phi) \tag{3.12}$$

for all $\varphi \in \mathcal{H}(\xi, \eta, \Phi, g)$.

Proof. By definition, we have

$$\begin{array}{rl} & V(I_{d\eta'}(\varphi + \phi) - I_{d\eta}(\varphi)) \\ = & \int_M \varphi((d\eta')^m - (d\eta)^m) \wedge \eta + \int_M \phi((d\eta')^m - (d\eta_\varphi)^m) \wedge \eta, \end{array} \tag{3.13}$$

where $d\eta_\varphi = d\eta + \sqrt{-1}\partial_B\bar{\partial}_B\varphi$. On the other hand, By direct calculation, we have

$$\begin{array}{rl} & |\int_M \varphi((d\eta')^m - (d\eta)^m) \wedge \eta| \\ = & |\int_M \varphi(-\sqrt{-1}\partial_B\bar{\partial}_B\phi) \wedge (\sum_{j=0}^{m-1}(d\eta')^j \wedge (d\eta)^{m-j-1}) \wedge \eta| \\ = & |\int_M \phi(-\sqrt{-1}\partial_B\bar{\partial}_B\varphi) \wedge (\sum_{j=0}^{m-1}(d\eta')^j \wedge (d\eta)^{m-j-1}) \wedge \eta| \\ = & |\int_M \phi(d\eta - d\eta_\varphi) \wedge (\sum_{j=0}^{m-1}(d\eta')^j \wedge (d\eta)^{m-j-1}) \wedge \eta| \\ \leq & mVOsc(\phi) \end{array} \tag{3.14}$$

and

$$|\int_M \phi((d\eta')^m - (d\eta_\varphi)^m) \wedge \eta| \leq VOsc(\phi). \tag{3.15}$$

Then (3.13), (3.14) and (3.15) imply (3.12).

□

Let ρ^T_φ denote the transverse Ricci form of the Sasakian structure $(\xi, \eta_\varphi, \Phi_\varphi, g_\varphi)$. $\int_M \rho^T_\varphi \wedge (d\eta_\varphi)^m \wedge \eta_\varphi$ is independent of the choice of $\varphi \in \mathcal{H}(\xi, \eta, \Phi, g)$ (e.g., Proposition 4.4 in [15]). This means that

$$\bar{S} = \frac{\int_M S^T_\varphi (d\eta_\varphi)^m \wedge \eta_\varphi}{\int_M (d\eta_\varphi)^m \wedge \eta_\varphi} = \frac{\int_M 2m\rho^T_\varphi \wedge (d\eta_\varphi)^{m-1} \wedge \eta}{\int_M (d\eta_\varphi)^m \wedge \eta}, \tag{3.16}$$

depends only on the basic Kähler class. As in the Kähler case (see [20]), we can define the Mabuchi's $\mathcal{K}$-energy on the space $\mathcal{H}(\xi, \eta, \Phi, g)$.

Definition 3.3. *Let φ' and φ'' are two basic functions in $\mathcal{H}(\xi, \eta, \Phi, g)$, we define*

$$\mathcal{M}(\varphi', \varphi'') := -\frac{1}{V}\int_0^1 \int_M \dot{\varphi}_t(S^T_t - \bar{S})(d\eta_t)^m \wedge \eta \, dt, \tag{3.17}$$

where φ_t ($t \in [0, 1]$) be a path in $\mathcal{H}$ connecting φ' and φ'', $\dot{\varphi}_t = \frac{\partial}{\partial t}\varphi_t$, S^T_t is the transverse scalar curvature to the Sasakian structure $(\xi, \eta_{\varphi_t}, \Phi_{\varphi_t}, g_{\varphi_t})$ and $\bar{S}$ is the average defined as in (3.16). We also define

$$\mathcal{V}_{d\eta_{\varphi'}}(\varphi) := \mathcal{M}(\varphi', \varphi' + \varphi) \tag{3.18}$$

for any $\varphi \in \mathcal{H}(\xi, \eta_{\varphi'}, \Phi_{\varphi'}, g_{\varphi'})$.

By Theorem 4.12 in [15] (or lemma 11 in [19]), we know that $\mathcal{M}$ is independent of the path φ_t, and so $\mathcal{M}$ is well defined. Furthermore, $\mathcal{M}$ satisfies the following cocycle condition, i.e.

$$\mathcal{M}(\varphi_0, \varphi_1) + \mathcal{M}(\varphi_1, \varphi_0) = 0, \tag{3.19}$$

$$\mathcal{M}(\varphi_0,\varphi_1)+\mathcal{M}(\varphi_1,\varphi_2)+\mathcal{M}(\varphi_2,\varphi_0)=0. \tag{3.20}$$

and

$$\mathcal{M}(\varphi_1+C',\varphi_2+C'')=\mathcal{M}(\varphi_1,\varphi_2) \tag{3.21}$$

for any $\varphi_i\in\mathcal{H}(\xi,\eta,\Phi,g)$ and $C',C''\in\mathbb{R}$. Following Ding [13], we also have the following relation between the functionals $\mathcal{V}_{d\eta}$ and $F_{d\eta}$.

Remark: *By the definitions and the above properties, it is easy to see that the functionals $F_{d\eta}$ and $\mathcal{V}_{d\eta}$ can also be defined on the space $\mathcal{S}(\xi,\bar{J})$.*

Lemma 3.4. Let (M,ξ,η,Φ,g) be a compact Sasakian manifold with $2\pi c_1^B(M)=(m+1)[d\eta]_B$, then

$$\begin{array}{ll} & \mathcal{V}_{d\eta}(\phi)-2(m+1)F_{d\eta}(\phi)\\ = & \frac{2}{V}\int_M h_{d\eta}(d\eta)^m\wedge\eta-\frac{2}{V}\int_M h_{d\eta_\phi}(d\eta_\phi)^m\wedge\eta \end{array} \tag{3.22}$$

for any $\phi\in\mathcal{H}(\xi,\eta,\Phi,g)$, where $h_{d\eta}$ and $h_{d\eta_\phi}$ are the normalized Ricci potential functions with respect to $d\eta$ and $d\eta_\phi$.

Proof. Let ϕ_t be a path connecting 0 with ϕ, by the definition, we have

$$\begin{array}{rcl} \mathcal{V}_{d\eta}(\phi) & = & -\frac{1}{V}\int_0^1\int_M\dot{\phi}_t(S_t^T-2m(m+1))(d\eta_t)^m\wedge\eta\,dt\\ & = & -\frac{2m}{V}\int_0^1\int_M\dot{\phi}_t(\rho_t^T-(m+1)d\eta_t)(d\eta_t)^{m-1}\wedge\eta\,dt\\ & = & -\frac{2m\sqrt{-1}}{V}\int_0^1\int_M\dot{\phi}_t\partial_B\bar{\partial}_B(h_{d\eta}-(m+1)\phi_t\\ & & -\log\frac{(d\eta_t)^m\wedge\eta}{(d\eta)^m\wedge\eta})(d\eta_t)^{m-1}\wedge\eta\,dt\\ & = & -\frac{2}{V}\int_0^1\int_M(h_{d\eta}-(m+1)\phi_t-\log\frac{(d\eta_t)^m\wedge\eta}{(d\eta)^m\wedge\eta})\frac{\partial}{\partial t}((d\eta_t)^m\wedge\eta)\,dt\\ & = & -2(m+1)(I_{d\eta}-J_{d\eta})(\phi)+\frac{2}{V}\int_M h_{d\eta}(d\eta^m-d\eta_\phi^m)\wedge\eta\\ & & +\frac{2}{V}\int_M\log\frac{(d\eta_\phi)^m\wedge\eta}{(d\eta)^m\wedge\eta}(d\eta_\phi)^m\wedge\eta. \end{array} \tag{3.23}$$

On the other hand, it is easy to check that

$$-\log\frac{(d\eta_\phi)^m\wedge\eta}{(d\eta)^m\wedge\eta}-(m+1)\phi+c=h_{d\eta_\phi}-h_{d\eta}, \tag{3.24}$$

where $c=-\log(\frac{1}{V}\int_M e^{h_{d\eta}-(m+1)\phi}d\eta^m\wedge\eta)$. Then

$$\begin{array}{rl} & \mathcal{V}_{d\eta}(\phi)=2(m+1)J_{d\eta}(\phi)-\frac{2(m+1)}{V}\int_M\phi d\eta^m\wedge\eta+2c\\ & +\frac{2}{V}\int_M h_{d\eta}(d\eta)^m\wedge\eta-\frac{2}{V}\int_M h_{d\eta_\phi}(d\eta_\phi)^m\wedge\eta\\ = & 2(m+1)F_{d\eta}(\phi)+\frac{2}{V}\int_M h_{d\eta}(d\eta)^m\wedge\eta-\frac{2}{V}\int_M h_{d\eta_\phi}(d\eta_\phi)^m\wedge\eta. \end{array}$$

□

By the normalization $\int_M\exp(h_{d\eta_\phi})(d\eta_\phi)^m\wedge\eta=V$, we known that $\int_M h_{d\eta_\phi}(d\eta_\phi)^m\wedge\eta\leq 0$, then we have the following corollary.

Corollary 3.5. Let (M,ξ,η,Φ,g) be a compact Sasakian manifold with $2\pi c_1^B(M)=(m+1)[d\eta]_B$, then

$$\mathcal{V}_{d\eta}(\phi)\geq 2(m+1)F_{d\eta}(\phi)+\frac{2}{V}\int_M h_{d\eta}(d\eta)^m\wedge\eta \tag{3.25}$$

for any $\phi\in\mathcal{H}(\xi,\eta,\Phi,g)$.

4. TRANSVERSE MONGE-AMPÈRE EQUATION

Let (M,ξ,η,Φ,g) be a compact Sasakian manifold with $2\pi c_1^B(M)=(m+1)[d\eta]_B$. Given any $\varphi\in\mathcal{H}(M,\xi,\eta,\Phi,g)$, we have a new Sasakian structure $(\xi,\eta_\varphi,\Phi_\varphi,g_\varphi)$. It is easy to check that $d\eta_\varphi$ is Sasakian-Einstein if and only if φ satisfies the following equation

$$\sqrt{-1}\partial_B\overline{\partial}_B\log\left(\frac{(d\eta_\varphi)^m\wedge\eta}{(d\eta)^m\wedge\eta}\right)=\sqrt{-1}\partial_B\overline{\partial}_B(h_{d\eta}-(m+1)\varphi), \tag{4.1}$$

which is equivalent to the transverse Monge-Ampère equation (1.5). As in Kähler case, we consider a family of equations (1.6). We set

$$S=\{t\in[0,1]|(1.6)\quad is\quad solvable\quad for\quad t\}. \tag{4.2}$$

By [14], we know that (1.6) is solvable for $t=0$. The openness of S was proved in [28](Proposition 5.3) and [25](Proposition 4.4) in a similar way as that in [1]. S is not empty. In order to use the continuity method to solve (1.5), we only need to prove the closedness of S. By El-Kacimi's ([14]) generalization of Yau's estimate ([35]) for transverse Monge-Ampère equations, the C^0-estimate for solutions of (1.6) implies the $C^{2,\alpha}$-estimate for them, and the transverse elliptic Schauder estimates give higher order estimates. Therefore it suffices to estimate C^0-norms of the solutions of (1.6). We list the following proposition for further discussion, the proof can be found in [28](proposition 5.3) and [25](proposition 4.4), see also proposition 4.2 in [37].

Proposition 4.1. *Let $0<\tau\leq 1$, and suppose that (1.6) has a solution φ_τ at $t=\tau$. If $0<\tau<1$, then there exists some $\epsilon>0$ such that φ_τ uniquely extends to a smooth family of solution $\{\varphi_t\}$ of (1.6) for $t\in(0,1)\cap(\tau-\epsilon,\tau+\epsilon)$. S is also open near $t=0$, i.e. there exists a small positive number ϵ such that there is a smooth family solution of (1.6) for $t\in(0,\epsilon)$. If (M,ξ,η,Φ,g) admits no nontrivial Hamiltonian holomorphic vector field, φ_1 can also be extended uniquely to a smooth family of solution $\{\varphi_t\}$ of (1.6) for $t\in(1-\epsilon,1]$.*

As in [4], we have the following lemma, the proof can be found in [37] (lemma 4.3), see also lemma 4.9 in [25] and lemma 5.4 in [28].

Lemma 4.2. Let $\{\varphi_t\}$ be a smooth family of solution of (1.6) for $t\in(0,1]$, then

$$\frac{d}{dt}(I_{d\eta}-J_{d\eta})(\varphi_t)\geq 0. \tag{4.3}$$

Now, we consider the existence problem of Sasakian-Einstein metrics. We prove the following theorem.

Theorem 4.3. *Let (M,ξ,η,Φ,g) be a compact Sasakian manifold with $[d\eta]_B=\frac{2\pi}{m+1}c_1^B(M,\mathcal{F}_\xi)$. If $F_{d\eta}$ (or $\mathcal{V}_{d\eta}$) is proper in the space $\mathcal{H}(\xi,\eta,\Phi,g)$, then there must exists a Sasakian-Einstein metric compatible with (ξ,η,Φ,g) .*

Proof. By Proposition 4.1, we can suppose that there exists a smooth family of solution $\{\varphi_t\}$ of (1.6) for $t\in(0,\tau)$ with some $\tau\in(0,1)$. From the equation (1.6), we know that $\triangle_t\varphi_t\leq 4m$ and $\rho^T_{d\eta_t}\geq t(m+1)d\eta_t$. By proposition 2.7, we have

$$\frac{1}{V}\int_M\varphi_t(d\eta_\varphi)^m\wedge\eta\leq inf_M\varphi_t+\frac{C_1(m)}{t}. \tag{4.4}$$

where positive constant $C_1(m)$ depends only on m. Using the fact $\triangle_{d\eta}\varphi_t \geq -4m$ and the Green formula, we have

$$\sup_M \varphi_t \leq \frac{1}{V}\int_M \varphi_t (d\eta)^m \wedge \eta + C_2 \tag{4.5}$$

where C_2 is a positive constant depends only on the geometry of (M, g). By the normalization, it's easy to check that $\sup_M \varphi_t \geq 0$ and $\inf_M \varphi_t \leq 0$. Then

$$\begin{aligned} \|\varphi_t\|_{C^0} &\leq \sup_M \varphi_t - \inf_M \varphi_t \\ &\leq I_{d\eta}(\varphi_t) + \tfrac{C_1(m)}{t} + C_2. \end{aligned} \tag{4.6}$$

By (3.8) and (4.3), we have

$$I_{d\eta}(\varphi_{t_1}) \leq (m+1)(I_{d\eta} - J_{d\eta})(\varphi_{t_2}) \tag{4.7}$$

for any $0 < t_1 \leq t_2 < \tau$. Combining (4.6) and (4.7), we get

$$t\|\varphi_t\|_{C^0} \leq t_0(m+1)(I_{d\eta} - J_{d\eta})(\varphi_{t_0}) + C_3 \tag{4.8}$$

for any $0 < t \leq t_0 < \tau$, where C_3 is a positive constant depends only on the geometry of (M, g). So, we obtain an uniform bound on $|\frac{(d\eta+\sqrt{-1}\partial_B\overline{\partial}_B\varphi_t)^m\wedge\eta}{(d\eta)^m\wedge\eta}|$ for $0 < t \leq t_0 < \tau$. By El-Kacimi's ([14]) generalization of Yau's C^0 estimate ([35]) for transverse Monge-Ampère equations, there exists an uniform constant C_4 such that

$$\|\varphi_t\|_{C^0} \leq C_4 \tag{4.9}$$

for $0 < t \leq t_0 < \tau$.

Differentiating (1.6) with respect to t, we have

$$\frac{1}{4}\triangle_t\dot{\varphi}_t = -t(m+1)\dot{\varphi}_s - (m+1)\varphi_t. \tag{4.10}$$

Using (3.5) and (4.10), we have

$$\begin{aligned} \tfrac{d}{dt}(tF^0_{d\eta}(\varphi_t)) &= F^0_{d\eta}(\varphi_t) - \tfrac{t}{V}\textstyle\int_M \dot{\varphi}_t (d\eta_\varphi)^m \wedge \eta \\ &= J_{d\eta}(\varphi_t) - \tfrac{1}{V}\textstyle\int_M \varphi_t (d\eta)^m \wedge \eta - \tfrac{t}{V}\int_M \dot{\varphi}_t (d\eta_\varphi)^m \wedge \eta \\ &= -(I_{d\eta}(\varphi_t) - J_{d\eta}(\varphi_t)) \leq 0. \end{aligned} \tag{4.11}$$

By the uniform C^0 estimate (4.9), it is easy to check that

$$tF^0_{d\eta}(\varphi_t) \to 0 \tag{4.12}$$

as $t \to 0$. So, from (4.11), we have

$$\begin{aligned} F_{d\eta}(\varphi_t) &\leq -\tfrac{1}{m+1}\log\{\tfrac{1}{V}\textstyle\int_M e^{h_{d\eta}-(m+1)\varphi}(d\eta)^m \wedge \eta\} \\ &\leq \tfrac{(1-t)}{V}\textstyle\int_M \varphi_t (d\eta_\varphi)^m \wedge \eta, \end{aligned} \tag{4.13}$$

where we have used the concavity of the logrithmic function. From (4.13) and (4.4), we have

$$F_{d\eta}(\varphi_t) \leq \frac{(1-t)}{t}C_1(m). \tag{4.14}$$

By (4.14) and (4.6), the properness of $F_{d\eta}$ implies that $J_{d\eta}(\varphi_t)$, and consequently, $\|\varphi_t\|_{C^0}$ is uniformly bounded for $t \in [\epsilon, \tau)$. Therefore, the equation (1.5) can be solved, i.e. there is a Sasakian-Einstein metric on M.

For the $\mathcal{K}$-energy case. It is easy to see that, along the solutions of (1.6), we have

$$S_t^T = 2(m+1)(m - \frac{(1-t)}{4}\triangle_{\varphi_t}\varphi_t), \tag{4.15}$$

and

$$\begin{array}{rcl} \mathcal{V}_{d\eta}(\varphi_t) & = & -2(m+1)(I_{d\eta} - J_{d\eta})(\varphi_t) + \frac{2}{V}\int_M h_{d\eta} d\eta^m \wedge \eta \\ & & -\frac{2t(m+1)}{V}\int_M \varphi_t (d\eta_{\varphi_t})^m \wedge \eta. \end{array} \tag{4.16}$$

Then, by (3.9) and (3.17), we have

$$\begin{array}{rcl} \frac{d}{dt}\mathcal{V}_{d\eta}(\varphi_t) & = & -\frac{1}{V}\int_M \dot{\varphi}_t (S_t^T - \bar{S})(d\eta_t)^m \wedge \eta \\ & = & \frac{2m+2}{V}\int_M \dot{\varphi}_t \frac{(1-t)}{4}\triangle_{\varphi_t}\varphi_t (d\eta_t)^m \wedge \eta \\ & = & 2(m+1)(t-1)\frac{d}{dt}((I_{d\eta} - J_{d\eta})(\varphi_t)). \end{array} \tag{4.17}$$

From (4.16) and (4.17), we have

$$\frac{d}{dt}(\frac{t}{V}\int_M \varphi_t (d\eta_{\varphi_t})^m \wedge \eta + t(I_{d\eta} - J_{d\eta})(\varphi_t)) = (I_{d\eta} - J_{d\eta})(\varphi_t). \tag{4.18}$$

Noting that $\frac{t}{V}\int_M \varphi_t (d\eta_{\varphi_t})^m \wedge \eta + t(I_{d\eta} - J_{d\eta})(\varphi_t) \to 0$ as $t \to 0$. The identity (4.18) implies that

$$\frac{1}{V}\int_M \varphi_t (d\eta_{\varphi_t})^m \wedge \eta + (I_{d\eta} - J_{d\eta})(\varphi_t) \geq 0, \tag{4.19}$$

and

$$\begin{array}{rcl} \mathcal{V}_{d\eta}(\varphi_t) & \leq & -2(m+1)(1-t)(I_{d\eta} - J_{d\eta})(\varphi_t) + \frac{2}{V}\int_M h_{d\eta} d\eta^m \wedge \eta \\ & \leq & \frac{2}{V}\int_M h_{d\eta} d\eta^m \wedge \eta. \end{array} \tag{4.20}$$

Then the properness of $\mathcal{V}_{d\eta}$ implies that $J_{d\eta}(\varphi_t)$, and consequently, $\|\varphi_t\|_{C^0}$ is uniformly bounded for $t \in [\epsilon, \tau)$. Therefore, the equation (1.5) can also be solved.

□

Proposition 4.4. *Let (M, ξ, η, Φ, g) be a compact Sasakian manifold with $[d\eta]_B = \frac{2\pi}{m+1}c_1^B(M, \mathcal{F}_\xi)$. If $\mathcal{V}_{d\eta}$(or $F_{d\eta}$) is bounded from below in the space $\mathcal{H}(\xi, \eta, \Phi, g)$, then there exists a smooth family of solution $\{\varphi_t\}$ of (1.6) for $t \in (0, 1)$.*

Proof. By corollary 3.5, it is sufficient to prove the $\mathcal{K}$-energy case. We prove it by contradiction. Let $\tau < 1$ be the maximal number such that there exists a smooth family of solution $\{\varphi_t\}$ of (1.6) for $t \in (0, \tau)$. By (4.17), we have

$$\begin{array}{rcl} (I_{d\eta} - J_{d\eta})(\varphi_t) & = & \int_{t_0}^t \frac{1}{2(m+1)(t-1)}\frac{d}{ds}\mathcal{V}_{d\eta}(\varphi_s) + (I_{d\eta} - J_{d\eta})(\varphi_{t_0}) \\ & \leq & \frac{1}{1-\tau}(V_{d\eta}(\varphi_{t_0}) - \inf V_{d\eta}(\varphi)) + +(I_{d\eta} - J_{d\eta})(\varphi_{t_0}), \end{array} \tag{4.21}$$

where $0 < t_0 < \tau$. Since $\mathcal{V}_{d\eta}$ is bounded from below, then $(I_{d\eta} - J_{d\eta})(\varphi_t)$ is bounded uniformly from above for $0 < t < \tau$, and consequently, $\|\varphi_t\|_{C^0}$ is uniformly bounded for $t \in [t_0, \tau)$. So the equation (1.6) can also be solved at τ, this gives a contradiction.

□

5. Smoothing by the Sasakian-Ricci flow

In this section, we use the Sasakian-Ricci flow to get a smoothing lemma. As a natural analogue of the Kähler-Ricci flow, the Sasakian-Ricci flow was introduced in [29]. Now, we consider the following Sasakian-Ricci flow

$$\frac{\partial v}{\partial s} = \log \frac{(d\tilde{\eta}_0 + \sqrt{-1}\partial_B\overline{\partial}_B v)^m \wedge \tilde{\eta}_0}{(d\tilde{\eta}_0)^m \wedge \tilde{\eta}_0} + (m+1)v - h_{d\tilde{\eta}_0}, \tag{5.1}$$

with $v|_{s=0} \equiv 0$. The long-time existence had been proved in [29]. In the following, for simplicity, we will denote the transverse Kähler form $d\tilde{\eta}_0 + \sqrt{-1}\partial_B\overline{\partial}_B v$ by $d\tilde{\eta}_s$, and we will use a subscript s to indicated objects that are defined with respect to the transverse Kähler metric $d\tilde{\eta}_s$. As that in [3], we have the following lemma.

Lemma 5.1. *The following inequalities*

$$\|\frac{\partial v}{\partial s}\|_{C0} \leq e^{(m+1)s}\|h_{d\tilde{\eta}_0}\|_{C^0}, \tag{5.2}$$

$$\sup_M(|h_{d\tilde{\eta}_s}|^2 + \frac{s}{2}|dh_{d\tilde{\eta}_s}|_s^2) \leq 4e^{2(m+1)s}\|h_{d\tilde{\eta}_0}\|^2_{C^0}, \tag{5.3}$$

$$e^{-(m+1)s}\triangle_s h_{d\tilde{\eta}_s} \geq \triangle_0 h_{d\tilde{\eta}_0}, \tag{5.4}$$

hold for all $s \geq 0$.

Proof. Differentiating the Sasakian-Ricci flow equation (5.1) gives

$$\frac{\partial}{\partial s}\dot{v} = \frac{1}{4}\triangle_s\dot{v} + (m+1)\dot{v}, \tag{5.5}$$

the maximum principle implies (5.2). By direct calculation, we have

$$\frac{\partial}{\partial s}|d\dot{v}|_s^2 = \frac{1}{4}\triangle_s|d\dot{v}|_s^2 - \frac{1}{2}|\nabla_s d\dot{v}|_s^2 + (m+1)|d\dot{v}|_s^2. \tag{5.6}$$

From (5.5) and (5.6), we have

$$(\frac{\partial}{\partial s} - \frac{1}{4}\triangle_s)(\dot{v}^2 + \frac{s}{2}|d\dot{v}|_s^2) \leq 2(m+1)(\dot{v}^2 + \frac{s}{2}|d\dot{v}|_s^2), \tag{5.7}$$

and the maximum principle implies that

$$\sup_M(\dot{v}^2 + \frac{s}{2}|d\dot{v}|_s^2) \leq e^{2(m+1)s}\|h_{d\tilde{\eta}_0}\|^2_{C^0}. \tag{5.8}$$

By the equation (5.1), it is easy to check that

$$h_{d\tilde{\eta}_s} = -\dot{v} + c_s \tag{5.9}$$

for some constant c_s with $c_0 = 0$. From the normalization condition the Ricci potential function and (5.1), we have

$$\int_M e^{-(m+1)v + h_{d\tilde{\eta}_0} + c_s}(d\tilde{\eta}_0)^m \wedge \tilde{\eta}_0 = \int_M e^{h_{d\tilde{\eta}_s}}(d\tilde{\eta}_s)^m \wedge \tilde{\eta}_s = V, \tag{5.10}$$

and then

$$|c_s| \leq (m+1)\|v\|_{C^0} \leq e^{(m+1)s}\|h_{d\tilde{\eta}_0}\|_{C^0}. \tag{5.11}$$

So, (5.8) and (5.11) imply (5.3).

By direct calculation, one can check that

$$(\frac{\partial}{\partial s} - \frac{1}{4}\triangle_s)(\triangle_s\dot{v}) = (m+1)\triangle_s\dot{v} - |\partial_B\bar{\partial}_B\dot{v}|_s^2, \tag{5.12}$$

and then the maximum principle implies the inequality (5.4).

□

Lemma 5.2. *Let $v_{t,s}$ be a solution of (5.1) with $d\tilde{\eta}_0 = d\eta_{\varphi_t}$. Let $\tilde{h} = h_{d\tilde{\eta}_1} - \frac{1}{V}\int_M h_{d\tilde{\eta}_1}(d\tilde{\eta}_1)^m \wedge \tilde{\eta}_1$ and assume that*

$$\frac{1}{2}d\eta_{SE} \leq d\tilde{\eta}_1 \leq d\eta_{SE}. \tag{5.13}$$

Then for any $p > 2m+1$, there exist positive constant $\bar{C}_1$ depending only on (M, g_{SE}) and p such that

$$\|\tilde{h}\|_{C^0} \leq \bar{C}_1(1-t)^{\frac{1}{p-1}}\|h_{d\tilde{\eta}_0}\|_{C^0}^{\frac{p-2}{p-1}}. \tag{5.14}$$

Proof. (5.3) implies that $\|\tilde{h}\|_{C^0} \leq 4e^{m+1}\|h_{d\tilde{\eta}_0}\|_{C^0}$. By the initial condition $d\tilde{\eta}_0 = d\eta_{\varphi_t}$, we have $\rho^T(d\tilde{\eta}_0) \geq t(m+1)d\tilde{\eta}_0$, and $\triangle_0 h_{d\tilde{\eta}_0} \geq 4m(m+1)(t-1)$. By (5.4), we have

$$-\triangle_1 h_{d\tilde{\eta}_1} \leq 4e^{m+1}m(m+1)(1-t). \tag{5.15}$$

Integrating by parts, we have

$$\begin{array}{rcl} \int_M |d\tilde{h}|_1^2(d\tilde{\eta}_1)^m \wedge \tilde{\eta}_1 & = & -\int_M \tilde{h}\triangle_1\tilde{h}(d\tilde{\eta}_1)^m \wedge \tilde{\eta}_1 \\ & \leq & \int_M(\tilde{h} - \inf\tilde{h})\sup_M(-\triangle_1\tilde{h})(d\tilde{\eta}_1)^m \wedge \tilde{\eta}_1 \\ & \leq & 2V\|\tilde{h}\|_{C^0}\sup_M(-\triangle_1\tilde{h}) \\ & \leq & \bar{C}_2(1-t)\|\tilde{h}\|_{C^0}, \end{array} \tag{5.16}$$

where $\bar{C}_2$ depends only on the dimension of M. Since $\tilde{h}$ be a basic function, by condition (5.13), we have

$$|d\tilde{h}|_{SE}^2 \leq 2|d\tilde{h}|_1^2 \tag{5.17}$$

Let $p > 2m+1$, by the Sobolev imbedding theorem (Lemma 2.22 of [2]), the Poincaré inequality and (5.3), we have

$$\begin{array}{rcl} \|\tilde{h}\|_{C^0}^p & \leq & \bar{C}_3(\int_M |\tilde{h}|^p + |d\tilde{h}|_{SE}^p(d\eta_{SE})^m \wedge \eta_{SE}) \\ & \leq & \bar{C}_4\|h_{d\tilde{\eta}_0}\|_{C^0}^{p-2}(\int_M |\tilde{h}|^2 + |d\tilde{h}|_{SE}^2(d\eta_{SE})^m \wedge \eta_{SE}) \\ & \leq & \bar{C}_5\|h_{d\tilde{\eta}_0}\|_{C^0}^{p-2}(\int_M |d\tilde{h}|_{SE}^2(d\eta_{SE})^m \wedge \eta_{SE}) \\ & \leq & \bar{C}_6\|h_{d\tilde{\eta}_0}\|_{C^0}^{p-2}(\int_M |d\tilde{h}|_1^2(d\tilde{\eta}_1)^m \wedge \tilde{\eta}_1), \end{array} \tag{5.18}$$

where constants $\bar{C}_i$ depends only on (M, g_{SE}) and p. Then (5.16) and (5.18) imply (5.14), and we are finished.

□

Lemma 5.3. *Let $v_{t,s}$ be a solution of (5.1) with initial data $d\tilde{\eta}_0 = d\eta_{\varphi_t}$, and $u_t = v_{t,1}$. We have the inequality*

$$\|u_t\|_{C^0} \leq \frac{1}{m+1}e^{m+1}\|h_{d\eta_{\varphi_t}}\|_{C^0} \tag{5.19}$$

for all $t \in [0,1]$. Moreover, assume that $\frac{1}{2}d\eta_{SE} \leq d\eta_{\varphi_t+u_t} \leq d\eta_{SE}$ for all $t \in [t_1, 1]$, where $t_1 \in [0,1)$. Then for any $p > 2m+1$ and $0 \leq k < 1$, there exists a constant $\bar{C}_7$ depending only on (M, g_{SE}) and p such that

$$\|h_{d\eta_{\varphi_t+u_t}}\|_{C^{0,k}(d\eta_{SE})} \leq \bar{C}_7(1-t)^{1-\beta}(1+\|h_{d\eta_{\varphi_t}}\|_{C^0})^\beta \tag{5.20}$$

for all $t \in [t_1, 1]$, where $\beta = \frac{p+k-2}{p-1}$.

Proof. From (5.2), it follows that $|\frac{\partial v_{t,s}}{\partial s}| \leq e^{(m+1)s}\|h_{d\eta_{\varphi_t}}\|_{C^0}$, and integrating from 0 to 1, we obtain the inequality (5.19).

In the following, let $d(x,y)$ be the distance between x and y with respect to the metric g_{SE}. Since $h_{d\eta_{\varphi_t+u_t}}$ is a basic function, by the condition $\frac{1}{2}d\eta_{SE} \leq d\eta_{\varphi_t+u_t} \leq d\eta_{SE}$, we have

$$|dh_{d\eta_{\varphi_t+u_t}}|_{d\eta_{SE}} \leq \sqrt{2}|dh_{d\eta_{\varphi_t+u_t}}|_{d\eta_{\varphi_t+u_t}}.$$

If $d(x,y) \leq (1-t)^{\frac{1}{p-1}}(1+\|h_{d\eta_{\varphi_t}}\|_{C^0})^{-\frac{1}{p-1}}$, by (5.3) in lemma 5.1, we have

$$\begin{aligned}
& |h_{d\eta_{\varphi_t+u_t}}(x) - h_{d\eta_{\varphi_t+u_t}}(y)| \leq d(x,y)\sup_M |dh_{d\eta_{\varphi_t+u_t}}|_{d\eta_{SE}} \\
\leq\ & \sqrt{2}d(x,y)\sup_M |dh_{d\eta_{\varphi_t+u_t}}|_{d\eta_{\varphi_t+u_t}} \\
\leq\ & 4\sqrt{2}e^{m+1}d(x,y)(1+\|h_{d\eta_{\varphi_t}}\|_{C^0}) \\
\leq\ & 4\sqrt{2}e^{m+1}(1-t)^{\frac{1-k}{p-1}}(1+\|h_{d\eta_{\varphi_t}}\|_{C^0})^{\frac{p+k-2}{p-1}}d(x,y)^k.
\end{aligned} \tag{5.21}$$

If $d(x,y) \geq (1-t)^{\frac{1}{p-1}}(1+\|h_{d\eta_{\varphi_t}}\|_{C^0})^{-\frac{1}{p-1}}$, then the estimate (5.14) in lemma 5.2 implies

$$\begin{aligned}
& |h_{d\eta_{\varphi_t+u_t}}(x) - h_{d\eta_{\varphi_t+u_t}}(y)| \leq 2\|\tilde{h}\|_{C^0} \\
\leq\ & 2\bar{C}_1(1-t)^{\frac{1}{p-1}}(\|h_{d\eta_{\varphi_t}}\|_{C^0})^{\frac{p-2}{p-1}} \\
\leq\ & 2\bar{C}_1(1-t)^{\frac{1-k}{p-1}}(1+\|h_{d\eta_{\varphi_t}}\|_{C^0})^{\frac{p+k-2}{p-1}}d(x,y)^k.
\end{aligned} \tag{5.22}$$

On the other hand, the integral normalization $\int_M e^{h_{d\eta_{\varphi_t+u_t}}}(d\eta_{\varphi_t+u_t})^m \wedge \eta = V$ implies $h_{d\eta_{\varphi_t+u_t}}$ change signs, so we have

$$\begin{aligned}
& \|h_{d\eta_{\varphi_t+u_t}}\|_{C^0} \leq Osc(h_{d\eta_{\varphi_t+u_t}}) = Osc(\tilde{h}) \leq 2\|\tilde{h}\|_{C^0} \\
\leq\ & 2\bar{C}_1(1-t)^{\frac{1}{p-1}}(\|h_{d\eta_{\varphi_t}}\|_{C^0})^{\frac{p-2}{p-1}}.
\end{aligned} \tag{5.23}$$

It is easy to see that (5.21), (5.22) and (5.23) imply the estimate (5.20).

□

Set $\alpha := 1 - \frac{1}{4m+2} > \frac{1}{2}$ and define the function $f_{d\eta}$ by

$$f_{d\eta}(t) := (1-t)^{1-\alpha}(1+2(1-t)\|\varphi_t\|_{C^0})^{\alpha}. \tag{5.24}$$

Discussing as that in [32], we have the following proposition.

Proposition 5.4. *Suppose that (M,ξ,η,Φ,g) admits no non-trivial Hamiltonian holomorphic vector fields. Let φ_t be a smooth family of solutions of the equation (1.6) for $t \in (0,1]$. There exist a constant $D > 0$ depending only on (M, g_{SE}) such that*

$$\|\varphi_1 - \varphi_t\|_{C^0} \leq A(1-t)\|\varphi_t\|_{C^0} + 1 \tag{5.25}$$

for all $t \in [t_0, 1]$, where $t_0 \in [0,1)$ satisfies $f_{d\eta}(t_0) = \max_{[t_0,1]} f_{d\eta} = D$ and A depending only on the dimension of M.

Proof. Let's rewrite (1.6) as the following transverse Monge-Ampère equation with $d\eta_{SE}$ as reference metric

$$\begin{aligned}
& \frac{(d\eta_{SE}+\sqrt{-1}\partial_B\overline{\partial}_B(\varphi_t-\varphi_1))^m\wedge\eta}{(d\eta_{SE})^m\wedge\eta} \\
& = \exp(-(m+1)(\varphi_t-\varphi_1)+(1-t)(m+1)\varphi_t).
\end{aligned} \tag{5.26}$$

It is easy to see that $h_{d\eta_{\varphi_t}} = (t-1)(m+1)\varphi_t + c_t$, for some constant c_t. The integrate normalization of the Ricci potential function $h_{d\eta_{\varphi_t}}$ gives

$$\begin{aligned}
V &= \int_M (d\eta_{\varphi_t})^m \wedge \eta = \int_M e^{h_{d\eta_{\varphi_t}}}(d\eta_{\varphi_t})^m \wedge \eta \\
&= \int_M e^{(t-1)(m+1)\varphi_t+c_t}(d\eta_{\varphi_t})^m \wedge \eta,
\end{aligned} \tag{5.27}$$

from which it follows that

$$|c_t| \leq (m+1)(1-t)\|\varphi_t\|_{C^0}, \tag{5.28}$$

and

$$\|h_{d\eta_{\varphi_t}}\|_{C^0} \leq 2(m+1)(1-t)\|\varphi_t\|_{C^0}. \tag{5.29}$$

Then, lemma 5.3 implies that

$$\|u_t\|_{C^0} \leq 2e^{(m+1)}(1-t)\|\varphi_t\|_{C^0}. \tag{5.30}$$

Consider $d\eta_{\varphi_t+u_t} = d\eta + \sqrt{-1}\partial_B\bar{\partial}_B(\varphi_t+u_t) = d\eta_{SE} + \sqrt{-1}\partial_B\bar{\partial}_B(\varphi_t+u_t-\varphi_1)$, and then

$$\begin{array}{l} \frac{(d\eta_{SE}+\sqrt{-1}\partial_B\overline{\partial}_B(\varphi_t+u_t-\varphi_1))^m\wedge\eta}{(d\eta_{SE})^m\wedge\eta} \\ = \exp(-(m+1)(\varphi_t+u_t-\varphi_1) - h_{d\eta_{\varphi_t+u_t}} - \tilde{c}_t) \end{array} \tag{5.31}$$

for some constant $\tilde{c}_t$. Setting $\tilde{\varphi}_t = \varphi_t + u_t - \varphi_1 + \frac{\tilde{c}_t}{m+1}$, from (5.31) and (5.30), we have

$$\begin{array}{rl} & \int_M e^{h_{d\eta_{\varphi_t+u_t}}}(d\eta_{\varphi_t+u_t})^m\wedge\eta = \int_M e^{-(m+1)\tilde{\varphi}_t}(d\eta_{SE})^m\wedge\eta \\ = & \int_M e^{-(m+1)\tilde{\varphi}_t+t(m+1)\varphi_t-(m+1)\varphi_1}(d\eta_{\varphi_t})^m\wedge\eta \\ = & \int_M e^{(t-1)(m+1)\varphi_t-(m+1)u_t-\tilde{c}_t}(d\eta_{\varphi_t})^m\wedge\eta, \end{array} \tag{5.32}$$

and then

$$\begin{array}{rcl} |\tilde{c}_t| & \leq & (1-t)(m+1)\|\varphi_t\|_{C^0} + (m+1)\|u_t\|_{C^0} \\ & \leq & (1-t)(m+1)(1+2e^{(m+1)})\|\varphi_t\|_{C^0}. \end{array} \tag{5.33}$$

Recall that $\varphi_t - \varphi_1 = \tilde{\varphi}_t - u_t - \frac{\tilde{c}_t}{m+1}$, from (5.30) and (5.33), we have

$$\|\varphi_t - \varphi_1\|_{C^0} = \|\tilde{\varphi}_t\|_{C^0} + (1-t)(4e^{(m+1)}+1)\|\varphi_t\|_{C^0}. \tag{5.34}$$

From above, it will suffice to get the estimate $\|\tilde{\varphi}_t\|_{C^0} \leq 1$.

Let's consider the following transverse Monge-Ampère equation

$$\log\{\frac{(d\eta_{SE}+\sqrt{-1}\partial_B\overline{\partial}_B\psi)^m\wedge\eta}{(d\eta_{SE})^m\wedge\eta}\} + (m+1)\psi = \tilde{\psi}. \tag{5.35}$$

The linearization of the left side of (5.35) at $\psi = 0$ is

$$\delta\psi \mapsto \frac{1}{4}\triangle_{SE}\delta\psi + (m+1)\delta\psi, \tag{5.36}$$

which is a transverse elliptic operator from $C_B^{i+2,k}(M) \to C_B^{i+2,k}(M)$ for any $0<k<1$ and $i \geq 0$. If M doesn't have non-trivial Hamiltonian holomorphic vector fields, by theorem 5.1 of [15], we have $ker(\frac{1}{4}\triangle_{SE}+(m+1)) = 0$, then the operator $(\frac{1}{4}\triangle_{SE}+(m+1)) : C_B^{i+2,\epsilon}(M) \to C_B^{i+2,\epsilon}(M)$ is invertible. Applying the implicit function theorem, there exist positive constants $\epsilon(d\eta_{SE})$ and $C^*(d\eta_{SE})$ which depend only on k and the geometry of (M, g_{SE}), so that

$$if \quad \|\tilde{\psi}\|_{C^{0,k}} \leq \epsilon(d\eta_{SE}) \quad then \quad \|\psi\|_{C^{2,k}} \leq C^*(d\eta_{SE})\|\tilde{\psi}\|_{C^{0,k}}. \tag{5.37}$$

Setting $D = \frac{\epsilon(m+1)^{-\alpha}}{2(\bar{C}_7+1)(C^*+1)(\epsilon+1)}$, where $\epsilon = \epsilon(d\eta_{SE})$, $C^* = C^*(d\eta_{SE})$ are chosen as in (5.37), $\alpha = 1 - \frac{1}{4m+2}$, $\bar{C}_7$ is defined as in lemma 5.3 (by choosing $k = \frac{1}{2}$ and $p = 2m+2$. Let $t_0 \in [0,1)$ satisfies $f_{d\eta}(t_0) = \max_{[t_0,1]} f_{d\eta} = D$. Now, we only need to prove the following claim.

Claim For all $t \in [t_0, 1]$, we have

$$\|\tilde{\varphi}_t\|_{C^{2,\frac{1}{2}}} < \frac{1}{2}. \tag{5.38}$$

We assume the contrary. Since $\tilde{\varphi}_1 = 0$, there exists $t_1 \in [t_0, 1)$ such that

$$\|\tilde{\varphi}_{t_1}\|_{C^{2,\frac{1}{2}}(d\eta_{SE})} = \frac{1}{2}, \quad \mathit{and} \quad \|\tilde{\varphi}_t\|_{C^{2,\frac{1}{2}}(d\eta_{SE})} < \frac{1}{2} \quad \mathit{if} \quad t_1 < t < 1. \tag{5.39}$$

In particular $-\frac{1}{4}d\eta_{SE} \leq \sqrt{-1}\partial_B\overline{\partial}_B\tilde{\varphi}_t \leq \frac{1}{4}d\eta_{SE}$, and then

$$\frac{3}{4}d\eta_{SE} \leq d\eta_{\varphi_t+u_t} \leq \frac{5}{4}d\eta_{SE} \tag{5.40}$$

for all $t \in [t_1, 1]$. By applying (5.20) in lemma 5.3 (by choosing $p = 2m + 2$) and (5.29), we have

$$\begin{aligned} \|h_{d\eta_{\varphi_t+u_t}}\|_{C^{0,\frac{1}{2}}(d\eta_{SE})} &\leq \bar{C}_7(1-t)^{1-\alpha}(1 + \|h_{d\eta_{\varphi_t}}\|_{C^0})^\alpha \\ &\leq \bar{C}_7(1-t)^{1-\alpha}(1 + 2(1-t)(m+1)\|\varphi_t\|_{C^0})^\alpha \\ &\leq \bar{C}_7(m+1)^\alpha(1-t)^{1-\alpha}(1 + 2(1-t)\|\varphi_t\|_{C^0})^\alpha \\ &\leq \bar{C}_7(m+1)^\alpha D \\ &= \tfrac{\bar{C}_7\epsilon}{2(\bar{C}_7+1)(C^*+1)(\epsilon+1)} \\ &< \epsilon, \end{aligned} \tag{5.41}$$

for all $t \in [t_1, 1]$. Using (5.37) again, we get

$$\begin{aligned} \|\tilde{\varphi}_{t_1}\|_{C^{2,\frac{1}{2}}(d\eta_{SE})} &\leq \|h_{d\eta_{\varphi_t+u_t}}\|_{C^{0,\frac{1}{2}}(d\eta_{SE})} \\ &\leq \tfrac{C^*\bar{C}_7\epsilon}{2(\bar{C}_7+1)(C^*+1)(\epsilon+1)} \\ &< \tfrac{1}{2}. \end{aligned} \tag{5.42}$$

This gives a contradiction, and complete the proof of the claim. So, the proof of the proposition is complete.

□

6. A Moser-Trudinger type inequality

In this section, we assume the existence of a Sasakian-Einstein structure and establish a Moser-Trudinger type inequality for functional $F_{d\eta_{SE}}$, our discussion follow that in [26] by Phong, Song, Strum and Weinkove. In fact, we obtain the following theorem.

Theorem 6.1. *Let $(M, \xi, \eta_{SE}, \Phi_{SE}, g_{SE})$ be a compact Sasakian-Einstein metric without non trivial Hamiltonian holomorphic vector field, then there exist uniform positive constants C_1, C_2 depending only the geometry of (M, g_{SE}), such that*

$$F_{d\eta_{SE}}(\varphi) \geq C_1 J_{d\eta_{SE}}(\varphi) - C_2, \tag{6.1}$$

for all $\varphi \in \mathcal{H}(\xi, \eta_{SE}, \Phi_{SE}, g_{SE})$.

Proof. Fix a basic function $\phi \in \mathcal{H}(\xi, \eta_E, \Phi_{SE}, g_{SE})$, and set $d\eta = d\eta_{SE} + \sqrt{-1}\partial_B\bar{\partial}_B\phi$. Now, let us consider the complex Monge-Ampère equation (1.6). Since there are no nontrivial Hamiltonian holomorphic vector fields, by the uniqueness of Sasakian-Einstein structure ([28] or [25]) and proposition 4.1, a unique solution φ_t exists for all $t \in (0, 1]$, and $d\eta_{\varphi_1} = d\eta_{SE}$. In particular φ_1 and $-\phi$ differ by a constant.

For further consideration, we give the following estimates for functionals F, I and J. From (3.1), (3.4) and (4.10), we have

$$
\begin{aligned}
\frac{d}{ds}(I_{d\eta}-J_{d\eta})(\varphi_s) &= -\frac{d}{ds}\Big(\frac{1}{V}\int_M \varphi_s(d\eta_{\varphi_s})^m\wedge\eta\Big)+\frac{1}{V}\int_M \dot\varphi_s(d\eta)^m\wedge\eta \\
&\quad -\frac{1}{V}\int_M \dot\varphi_s\{(d\eta)^m-(d\eta_{\varphi_s})^m\}\wedge\eta \\
&= -\frac{d}{ds}\Big(\frac{1}{V}\int_M \varphi_s(d\eta_{\varphi_s})^m\wedge\eta\Big)+\frac{1}{V}\int_M \dot\varphi_s(d\eta_{\varphi_s})^m\wedge\eta \\
&= -\frac{d}{ds}\Big(\frac{1}{V}\int_M \varphi_s(d\eta_{\varphi_s})^m\wedge\eta\Big)-\frac{1}{sV}\int_M \varphi_s(d\eta_{\varphi_s})^m\wedge\eta.
\end{aligned}
\tag{6.2}
$$

The uniform C^0 estimate (4.9) of φ_t implies that

$$
s\frac{1}{V}\int_M \varphi_s(d\eta_{\varphi_s})^m\wedge\eta\to 0
\tag{6.3}
$$

as $s\to 0$. By integrating on $[0,t]$, we get

$$
\begin{aligned}
&t(I_{d\eta}-J_{d\eta})(\varphi_t)-\int_0^t(I_{d\eta}-J_{d\eta})(\varphi_s)ds \\
=\ & \int_0^t s\frac{d}{ds}(I_{d\eta}-J_{d\eta})(\varphi_s)ds \\
=\ & -\int_0^t s\frac{d}{ds}\Big(\frac{1}{V}\int_M \varphi_s(d\eta_{\varphi_s})^m\wedge\eta\Big)ds-\frac{1}{V}\int_0^t\Big(\int_M \varphi_s(d\eta_{\varphi_s})^m\wedge\eta\Big)ds \\
=\ & -\frac{t}{V}\int_M \varphi_t(d\eta_{\varphi_t})^m\wedge\eta,
\end{aligned}
\tag{6.4}
$$

and then

$$
\begin{aligned}
F^0_{d\eta}(\varphi_t) &= -(I_{d\eta}-J_{d\eta})(\varphi_t)-\frac{1}{V}\int_M \varphi_t(d\eta_{\varphi_t})^m\wedge\eta \\
&= \frac{-1}{t}\int_0^t(I_{d\eta}-J_{d\eta})(\varphi_s)ds.
\end{aligned}
\tag{6.5}
$$

Taking $t=1$ and considering $F_{d\eta}(\varphi_1)=-F_{d\eta_{SE}}(\phi)$, so that

$$
F_{d\eta_{SE}}(\phi)=\int_0^1(I_{d\eta}-J_{d\eta})(\varphi_s)ds.
\tag{6.6}
$$

By the definitions (3.1) and the cocycle property of $F^0_{d\eta}$, we have

$$
\begin{aligned}
J_{d\eta}(\varphi_1)-J_{d\eta}(\varphi_t) &= \frac{1}{V}\int_M(\varphi_1-\varphi_t)(d\eta)^m\wedge\eta+F^0_{d\eta}(\varphi_1)-F^0_{d\eta}(\varphi_t) \\
&= \frac{1}{V}\int_M(\varphi_1-\varphi_t)(d\eta)^m\wedge\eta-F^0_{d\eta_{\varphi_1}}(\varphi_t-\varphi_1) \\
&\leq \frac{1}{V}\int_M(\varphi_1-\varphi_t)(d\eta)^m\wedge\eta+\frac{1}{V}\int_M(\varphi_t-\varphi_1)(d\eta_{\varphi_1})^m\wedge\eta) \\
&\leq Osc(\varphi_1-\varphi_t).
\end{aligned}
\tag{6.7}
$$

By adding

$$
\begin{aligned}
&I_{d\eta}(\varphi_t)-I_{d\eta}(\varphi_1) \\
=\ & \frac{1}{V}\int_M(\varphi_t-\varphi_1)(d\eta)^m\wedge\eta+\frac{1}{V}\int_M(\varphi_1-\varphi_t)(d\eta^m_{\varphi_1})\wedge\eta \\
&+\frac{1}{V}\int_M \varphi_t\{(d\eta_{\varphi_1})^m-(d\eta_{\varphi_t})^m\}\wedge\eta,
\end{aligned}
$$

we get

$$
\begin{aligned}
&(I_{d\eta}-J_{d\eta})(\varphi_t)-(I_{d\eta}-J_{d\eta})(\varphi_1) \\
=\ & J_{d\eta}(\varphi_1)-J_{d\eta}(\varphi_t)+(I_{d\eta}(\varphi_t)-I_{d\eta}(\varphi_1)) \\
\leq\ & \frac{1}{V}\int_M \varphi_t\{(\omega_{\varphi_1})^m-(\omega_{\varphi_t})^m\} \\
=\ & \frac{1}{V}\int_M \varphi_t(d\eta_{\varphi_1}-d\eta_{\varphi_t})\wedge\Big(\sum_{j=0}^{m-1} d\eta^j_{\varphi_t}\wedge d\eta^{(m-1-j)}_{\varphi_1}\Big)\wedge\eta \\
=\ & \frac{1}{V}\int_M(\varphi_1-\varphi_t)(d\eta_{\varphi_t}-d\eta)\wedge\Big(\sum_{j=0}^{m-1} d\eta^j_{\varphi_t}\wedge d\eta^{(m-1-j)}_{\varphi_1}\Big)\wedge\eta \\
\leq\ & mOsc(\varphi_1-\varphi_t).
\end{aligned}
\tag{6.8}
$$

Interchanging φ_t and φ_1 in (6.7) and (6.8), we get

$$
|J_{d\eta}(\varphi_1)-J_{d\eta}(\varphi_t)|\leq Osc(\varphi_1-\varphi_t)
\tag{6.9}
$$

and

$$
|(I_{d\eta}-J_{d\eta})(\varphi_t)-(I_{d\eta}-J_{d\eta})(\varphi_1)|\leq m\cdot Osc(\varphi_1-\varphi_t).
\tag{6.10}
$$

Using the relationship $F_{d\eta}(\varphi_1)=-F_{d\eta_{SE}}(\phi)$, we have

$$
\begin{aligned}
J_{d\eta}(\varphi_1) &= F_{d\eta}(\varphi_1)+\frac{1}{V}\int_M \varphi_1(d\eta)^m\wedge\eta\\
&= -F_{d\eta_{SE}}(\phi)+\frac{1}{V}\int_M \varphi_1(d\eta)^m\wedge\eta\\
&= -J_{d\eta_{SE}}(\phi)+\frac{1}{V}\int_M \phi\{(d\eta_{SE})^m-(d\eta)^m\wedge\eta\}\\
&= (I_{d\eta_{SE}}-J_{d\eta_{SE}})(\phi)\geq \frac{1}{m}J_{d\eta_{SE}}(\phi),
\end{aligned}
\tag{6.11}
$$

where we have used the inequality (3.8). Since $(I_{d\eta}-J_{d\eta})(\varphi_t)$ is nondecreasing in t, (6.6) implies that

$$
F_{d\eta_{SE}}(\phi)\geq (1-t)(I_{d\eta}-J_{d\eta})(\varphi_t)ds\geq \frac{1-t}{m}J_{d\eta}(\varphi_t),
\tag{6.12}
$$

using (6.11) and (6.7), we have

$$
F_{d\eta_{SE}}(\phi)\geq \frac{1-t}{m^2}J_{d\eta_{SE}}(\phi)-\frac{1-t}{m}Osc(\varphi_t-\varphi_1).
\tag{6.13}
$$

In the following, we choose t_0 as that in proposition 5.4. If $2(1-t_0)\|\varphi_{t_0}\|_{C^0}\leq 1$, by the definition of t_0 , we have $D\leq (1-t_0)^{1-\alpha}2^{\alpha}$, i.e.

$$
(1-t_0)\geq 2^{-\frac{\alpha}{1-\alpha}}D^{\frac{1}{1-\alpha}}.
\tag{6.14}
$$

If $2(1-t_0)\|\varphi_{t_0}\|_{C^0}\geq 1$, we have $D\leq 4^{\alpha}(1-t_0)\|\varphi_t\|^{\alpha}_{C^0}$, then

$$
(1-t_0)\geq \frac{D}{4^{\alpha}\|\varphi_{t_0}\|^{\alpha}_{C^0}}.
\tag{6.15}
$$

On the second case, we may assume that $1-t_0<\frac{A^{-1}}{2}$, the inequality implies that

$$
\|\varphi_{t_0}\|_{C^0}\leq 2\|\varphi_1\|_{C^0}+2,
\tag{6.16}
$$

then

$$
(1-t_0)\geq \frac{D}{4^{\alpha}(2\|\varphi_1\|_{C^0}+2)^{\alpha}}.
\tag{6.17}
$$

Since $\sup\varphi_1\cdot\inf\varphi_1\leq 0$, we always have the following inequality

$$
\begin{aligned}
(1-t_0) &\geq \frac{C'}{(\|\varphi_1\|_{C^0}+1)^{\alpha}}\\
&\geq \frac{C'}{(Osc(\varphi_1)+1)^{\alpha}},\\
&= \frac{C'}{(Osc(\phi)+1)^{\alpha}},
\end{aligned}
\tag{6.18}
$$

where C' is a positive constant depending only on (M,g_{SE}). On the other hand, using proposition 5.4 again, we have

$$
\begin{aligned}
(1-t_0)\|\varphi_1-\varphi_{t_0}\|_{C^0} &\leq (1-t_0)^2A\|\varphi_{t_0}\|_{C^0}+1\\
&\leq AD^{\frac{1}{\alpha}}+1.
\end{aligned}
\tag{6.19}
$$

By inequalities (6.13), (6.18) and (6.19), we obtain

$$
F_{d\eta_{SE}}(\phi)\geq \tilde{C}_1\frac{J_{d\eta_{SE}}(\phi)}{(Osc(\phi)+1)^{\alpha}}-\tilde{C}_2,
\tag{6.20}
$$

for all $\phi\in\mathcal{H}(\xi,\eta_{SE},\Phi_{SE},g_{SE})$, where $\tilde{C}_1$ and $\tilde{C}_2$ are positive constants depending only on the geometry of (M,g_{SE}).

Since $\varphi_t-\varphi_1\in\mathcal{H}(\xi,\eta_{SE},\Phi_{SE},g_{SE})$ and $\rho^T_{d\eta_t}\geq t(m+1)d\eta_t$, we can use corollary 2.8 to obtain the following estimate

$$
Osc(\varphi_t-\varphi_1)\leq I_{d\eta_{SE}}(\varphi_t-\varphi_1)+\bar{C}(M,g_{SE}),
\tag{6.21}
$$

for $t \in [\frac{1}{2}, 1]$, where $\bar{C}(M, g_{SE})$ is a constant depending only on (M, g_{SE}). By (6.20) and (6.21), we have

$$F_{d\eta_{SE}}(\varphi_t - \varphi_1) \geq \tilde{C}_3 \frac{J_{d\eta_{SE}}(\varphi_t - \varphi_1)}{(J_{d\eta_{SE}}(\varphi_t - \varphi_1) + 1)^{\alpha}} - \tilde{C}_2, \tag{6.22}$$

for $t \in [\frac{1}{2}, 1]$, where $\tilde{C}_3$ is a constant depending only on (M, g_{SE}).

By the cocycle property of the functional F, formulas (6.4), (6.5), (4.6), nondecreasing of $(I_{d\eta} - J_{d\eta})(\varphi_t)$ and the concavity of the log function, we have

$$\begin{array}{ll} & F_{d\eta_{SE}}(\varphi_t - \varphi_1) = F_{d\eta}(\varphi_t) - F_{d\eta}(\varphi_1) \\ = & \frac{-1}{t}\int_0^t (I_{d\eta} - J_{d\eta})(\varphi_s)ds + \int_0^1 (I_{d\eta} - J_{d\eta})(\varphi_s)ds \\ & -\frac{1}{m+1}\log\{\frac{1}{V}\int_M e^{(t-1)(m+1)\varphi_t}(d\eta_{\varphi_t})^m \wedge \eta\} \\ \leq & \frac{t-1}{t}\int_0^t (I_{d\eta} - J_{d\eta})(\varphi_s)ds + \int_t^1 (I_{d\eta} - J_{d\eta})(\varphi_s)ds \\ & +\frac{(1-t)}{V}\int_M \varphi_t (d\eta_{\varphi_t})^m \wedge \eta \\ = & \int_t^1 (I_{d\eta} - J_{d\eta})(\varphi_s)ds - (1-t)(I_{d\eta} - J_{d\eta})(\varphi_t) \\ \leq & (1-t)\{(I_{d\eta} - J_{d\eta})(\varphi_1) - (I_{d\eta} - J_{d\eta})(\varphi_t)\} \\ \leq & m(1-t)Osc(\varphi_1 - \varphi_t) \\ \leq & m(1-t)\{I_{d\eta_{SE}}(\varphi_t - \varphi_1) + \frac{C_1(m)}{t} + C_2\} \\ \leq & m(1-t)\{(m+1)J_{d\eta_{SE}}(\varphi_t - \varphi_1) + \frac{C_1(m)}{t} + C_2\} \end{array} \tag{6.23}$$

By a same discussion in [26] (p1083), we know that (6.13), (6.21), (6.22) and (6.23) imply the Moser-Trudinger inequality (6.1). We write out the proof in details just for reader's convenience.

Combining (6.22) with (6.23), we have

$$m(m+1)(1-t)J(t) + \tilde{C}_4(1-t) \geq \tilde{C}_3 \frac{J(t)}{(J(t)+1)^{\alpha}} - \tilde{C}_2, \tag{6.24}$$

for $t \in [\frac{1}{2}, 1]$, where $\tilde{C}_4$ is a constant depending only on (M, g_{SE}). Here we denote $J_{d\eta_{SE}}(\varphi_t - \varphi_1)$ by $J(t)$ just for simplicity. (6.24) can also be written as

$$\frac{J(t)}{(J(t)+1)^{\alpha}}(\tilde{C}_5 - (1-t)(J(t)+1)^{\alpha}) \leq \tilde{C}_6(1-t) + \tilde{C}_7 \tag{6.25}$$

where $\tilde{C}_5$, $\tilde{C}_6$ and $\tilde{C}_7$ are constants depending only on (M, g_{SE}). We can suppose that there exists a $t' \in [\frac{1}{2}, 1]$ with

$$(1-t')(J(t')+1)^{\alpha} = \frac{1}{2}\tilde{C}_5. \tag{6.26}$$

If not then we must have $(1-t)(J(t)+1)^{\alpha} < \frac{1}{2}\tilde{C}_5$ for all $t \in [\frac{1}{2}, 1]$. It would follow that $J(\frac{1}{2}) \leq \tilde{C}_5^{\frac{1}{\alpha}}$, then (6.13) and (6.21) imply (6.1). Otherwise, from (6.25) we have that $J(t') \leq \tilde{C}_8$ and $1 - t' \geq \tilde{C}_9$, these also imply (6.1).

□

Now, theorem 4.3 and theorem 6.1 imply the main theorem in the introduction.

7. A Miyaoka-Yau type inequality

Definition 7.1. *Let (M, ξ, η, Φ, g) be a compact Sasakian manifold with $[d\eta]_B = \frac{2\pi}{m+1}c_1^B(M, \mathcal{F}_\xi)$. As above, we define $\mathcal{S}(\xi, \bar{J})$ to the space of all Sasakian structures which compatible with (ξ, η, Φ, g). Let's define two positive constants by*

$$\alpha(\xi, \bar{J}) := \inf\{\lambda \quad | \quad 0 \leq S^T_{d\eta'} \leq 2m\lambda \quad for \quad some \quad (\xi, \eta, \Phi, g) \in \mathcal{S}(\xi, \bar{J})\};$$

and

$$\beta(\xi,\bar{J}) := \sup\{\lambda \quad | \quad S^T_{d\eta'} \geq 2m\lambda \quad \textit{for} \quad \textit{some} \quad (\xi,\eta,\Phi,g)\in\mathcal{S}(\xi,\bar{J})\}.$$

Remark: *Since the mean value of transverse Scalar curvature $\bar{S}=2m(m+1)$ for any Sasakian structure in $\mathcal{S}(\xi,\bar{J})$, it is easy to see that $\alpha(\xi,\bar{J})\geq m+1$ and $0<\beta(\xi,\bar{J})\leq m+1$. Obviously, if there exists a Sasakian-Einstein structure in $\mathcal{S}(\xi,\bar{J})$, then we have $\alpha(\xi,\bar{J})=m+1=\beta(\xi,\bar{J})$.*

Lemma 7.2. *Let Let (M,ξ,η,Φ,g) be a compact Sasakian manifold with $[d\eta]_B=\frac{2\pi}{m+1}c_1^B(M,\mathcal{F}_\xi)$, and $(\xi,\eta',\Phi',g')\in\mathcal{S}(\xi,\bar{J})$. Then we have*

$$\begin{array}{rl}(7.1) & \int_M (2\pi)^2(2c_2^B(M,\mathcal{F}_\xi)-\frac{m}{m+1}c_1^B(M,\mathcal{F}_\xi)^2)\wedge\frac{(\frac{1}{2}d\eta')^{m-2}}{(m-2)!}\wedge\eta' \\ = & \int_M |Rm^T|^2-\frac{2(S^T)^2}{m(m+1)}-\frac{(m-1)(m+2)}{m(m+1)}((S^T)^2-(2m(m+1))^2)\frac{(\frac{1}{2}d\eta')^m}{m!}\wedge\eta',\end{array}$$

where Rm^T and S^T are the transverse curvature tensor and the transverse scalar curvature of (ξ,η',Φ',g').

Proof. By direct calculation, we have

$$\begin{array}{rl} & \int_M (2\pi)^2(2c_2^B(M,\mathcal{F}_\xi)-\frac{m}{m+1}c_1^B(M,\mathcal{F}_\xi)^2)\wedge\frac{(\frac{1}{2}d\eta')^{m-2}}{(m-2)!}\wedge\eta' \\ = & \int_M\{tr(Rm^T\wedge Rm^T)-\frac{1}{m+1}trRm^T\wedge trRm^T\}\wedge\frac{(\frac{1}{2}d\eta')^{m-2}}{(m-2)!}\wedge\eta' \\ = & \int_M |Rm^T|^2-|\rho^T|^2+\frac{1}{m+1}((S^T)^2-|\rho^T|^2)\frac{(\frac{1}{2}d\eta')^m}{m!}\wedge\eta' \\ = & \int_M |Rm^T|^2-(S^T)^2-\frac{m+2}{m+1}(|\rho^T|^2-(S^T)^2)\frac{(\frac{1}{2}d\eta')^m}{m!}\wedge\eta'.\end{array}\tag{7.2}$$

On the other hand

$$\begin{array}{rl} & \int_M (S^T)^2-|\rho^T|^2\frac{(\frac{1}{2}d\eta')^m}{m!}\wedge\eta' \\ = & \int_M \rho^T\wedge\rho^T\frac{(\frac{1}{2}d\eta')^{m-2}}{(m-2)!}\wedge\eta' \\ = & \int_M 4m(m-1)(m+1)^2\frac{(\frac{1}{2}d\eta')^m}{m!}\wedge\eta'.\end{array}\tag{7.3}$$

Combining the above two equalities, we get (7.1).

□

In locally foliation chart $(x,z^1,\cdots,z^m)$, setting

$$Q_{i\bar{j}k\bar{l}}=R^T_{i\bar{j}k\bar{l}}-\frac{S^T}{m(m+1)}(g^T_{i\bar{j}}g^T_{k\bar{l}}+g^T_{i\bar{l}}g^T_{k\bar{j}}).\tag{7.4}$$

It is easy to check that

$$|Q|^2=|Rm^T|^2-\frac{2(S^T)^2}{m(m+1)}.\tag{7.5}$$

Combining (7.1) and (7.5), we have

$$\begin{array}{rl} & \int_M (2\pi)^2(2c_2^B(M,\mathcal{F}_\xi)-\frac{m}{m+1}c_1^B(M,\mathcal{F}_\xi)^2)\wedge\frac{(\frac{1}{2}d\eta')^{m-2}}{(m-2)!}\wedge\eta' \\ \geq & \int_M -\frac{(m-1)(m+2)}{m(m+1)}((S^T)^2-(2m(m+1))^2)\frac{(\frac{1}{2}d\eta')^m}{m!}\wedge\eta'.\end{array}\tag{7.6}$$

Let's recall the Calabi functional on the space $\mathcal{S}(\xi,\bar{J})$, which was introduce by Boyer, Galicki and Simanca in [11],

$$\begin{array}{rcl}\mathcal{C}al(\xi,\eta',\Phi',g') & = & \int_M (S^T_{d\eta'}-2m(m+1))^2(d\eta')^m\wedge\eta' \\ & = & \int_M (S^T_{d\eta'})^2-(2m(m+1))^2(d\eta')^m\wedge\eta'.\end{array}\tag{7.7}$$

If $\inf_{\mathcal{S}(\xi,\bar{J})} \mathcal{C}al = 0$, for arbitrary $\epsilon > 0$, we have a Sasakian structure $(\xi, \eta', \Phi', g') \in \mathcal{S}(\xi, \bar{J})$ such that $\mathcal{C}al(\xi, \eta', \Phi', g') \leq \epsilon$. Then, by (7.6), we have

$$\begin{array}{rl} & \int_M (2\pi)^2 (2c_2^B(M, \mathcal{F}_\xi) - \frac{m}{m+1} c_1^B(M, \mathcal{F}_\xi)^2) \wedge \frac{(\frac{1}{2} d\eta')^{m-2}}{(m-2)!} \wedge \eta' \\ \geq & -\frac{(m-1)(m+2)}{m(m+1)} \epsilon. \end{array} \tag{7.8}$$

Since ϵ is arbitrary, (7.8) implies the following theorem.

Theorem 7.3. *Let (M, ξ, η, Φ, g) be a compact Sasakian manifold with $[d\eta]_B = \frac{2\pi}{m+1} c_1^B(M, \mathcal{F}_\xi)$. If $\inf_{\mathcal{S}(\xi,\bar{J})} \mathcal{C}al = 0$, then we have the following Miyaoka-Yau type inequality*

$$\int_M (2c_2^B(M, \mathcal{F}_\xi) - \frac{m}{m+1} c_1^B(M, \mathcal{F}_\xi)^2) \wedge (d\eta)^{m-2} \wedge \eta \geq 0. \tag{7.9}$$

On the other hand, if $\alpha(\xi, \bar{J}) = m+1$, for arbitrary $\epsilon > 0$, we have a Sasakian structure $(\xi, \eta', \Phi', g') \in \mathcal{S}(\xi, \bar{J})$ such that $0 \leq S^T \leq 2m(m+1+\epsilon)$. By (7.7), we have

$$\mathcal{C}al(\xi, \eta', \Phi', g') \leq 2(2m)^2(m+1)\epsilon + (2m)^2\epsilon^2. \tag{7.10}$$

Then, we have the following corollary.

Corollary 7.4. *Let (M, ξ, η, Φ, g) be a compact Sasakian manifold with $[d\eta]_B = \frac{2\pi}{m+1} c_1^B(M, \mathcal{F}_\xi)$. If $\alpha(\xi, \bar{J}) = m+1$, then $\inf_{\mathcal{S}(\xi,\bar{J})} \mathcal{C}al = 0$. In particulary, we also have the Miyaoka-Yau type inequality (7.9).*

As that in [3], we have the following proposition.

Proposition 7.5. *Let (M, ξ, η, Φ, g) be a compact Sasakian manifold with $[d\eta]_B = \frac{2\pi}{m+1} c_1^B(M, \mathcal{F}_\xi)$. If the $\mathcal{K}$ energy functional $\mathcal{V}_{d\eta}$ is bounded below on the space $\mathcal{H}(\xi, \eta, \Phi, g)$, then, for arbitrary $\epsilon > 0$, M admits a Sasakian structure (ξ, η', Φ', g') compatible with (ξ, η, Φ, g) such that $|S^T_{d\eta'} - 2m(m+1)| \leq \epsilon$. In particularly, $\alpha(\xi, \bar{J}) = m+1 = \beta(\xi, \bar{J})$.*

Proof. By proposition 4.4, there exists a smooth family of solution $\{\varphi_t\}$ of (1.6) for $t \in (0,1)$. Let $f(t) = (1-t)(I_{d\eta} - J_{d\eta})(\varphi_t)$, by (4.17), we have

$$\frac{d}{dt} f(t) + (1-t)^{-1} f(t) = \frac{-1}{2(m+1)} \frac{d}{dt} \mathcal{V}_{d\eta}(\varphi_t). \tag{7.11}$$

Since $\mathcal{V}_{d\eta}$ is bounded below, the above equality implies that there exists a sequence $t_i \to 1$ such that $f(t_i) \to 0$ as $i \to +\infty$. From (4.6) and (3.8), we have

$$\begin{array}{rcl} \|h_{d\eta_t}\|_{C^0} & \leq & Osc(h_{d\eta_t}) = (1-t) Osc(\varphi_t) \\ & \leq & (1-t)((m+1)(I_{d\eta} - J_{d\eta})(\varphi_t) + \frac{C_1(m)}{t} + C_2). \end{array} \tag{7.12}$$

So, there exists a sequence $t_i \to 1$ such that $\|h_{d\eta_{t_i}}\|_{C^0} \to 0$ as $i \to +\infty$. On the other hand, considering

$$\rho^T_{d\eta_t} = t(m+1) d\eta_t + (m+1)(1-t) d\eta \geq t(m+1) d\eta_t, \tag{7.13}$$

for arbitrary $\epsilon > 0$, we get a Sasakian structure $(\xi, \tilde{\eta}, \tilde{\Phi}, \tilde{g}) \in \mathcal{S}(\xi, \bar{J})$ such that $S^T_{d\tilde{\eta}} - 2m(m+1) \geq -\epsilon$ and $\|h_{d\tilde{\eta}}\|_{C^0} < \epsilon$.

Let's consider the Sasakian-Ricci flow (5.1) with the initial data $d\tilde{\eta}_0 = d\tilde{\eta}$. Since the initial $h_{d\tilde{\eta}_0}$ satisfies $\|h_{d\tilde{\eta}_0}\|_{C^0} < \epsilon$ and $\triangle_0 h_{d\tilde{\eta}_0} \geq -2\epsilon$, by lemma 5.1, we have

$$\|h_{d\tilde{\eta}_s}\|_{C^0} < 4e^{2(m+1)}\epsilon, \quad for \quad s \in [0,2]; \tag{7.14}$$

$$\sup_M |dh_{d\tilde{\eta}_s}|_s^2 < 8e^{4(m+1)}\epsilon^2, \quad for \quad s\in[1,2]; \tag{7.15}$$

and

$$\triangle_s h_{d\tilde{\eta}_s} \geq -2e^{2(m+1)}\epsilon, \quad for \quad s\in[0,2]. \tag{7.16}$$

From (5.6) and (5.12), setting $a=\frac{1}{4m}$, we have

$$\begin{aligned} & (\tfrac{\partial}{\partial s}-\tfrac{1}{4}\triangle_s)(|dh_{d\tilde{\eta}_s}|_s+\epsilon a(s-1)\triangle_s h_{d\tilde{\eta}_s}) \\ \leq\ & (m+1)(|dh_{d\tilde{\eta}_s}|_s+\epsilon a(s-1)\triangle_s h_{d\tilde{\eta}_s})+\epsilon a\triangle_s h_{d\tilde{\eta}_s} \\ & -(1+\epsilon a(s-1))|\partial_B\bar{\partial}_B h_{d\tilde{\eta}_s}|_s^2 \\ \leq\ & (m+1)(|dh_{d\tilde{\eta}_s}|_s+\epsilon a(s-1)\triangle_s h_{d\tilde{\eta}_s})+\triangle_s h_{d\tilde{\eta}_s}(\epsilon a-\tfrac{1+\epsilon a(s-1)}{4m}\triangle_s h_{d\tilde{\eta}_s}) \end{aligned} \tag{7.17}$$

where we have used the Cauchy-Schwarz inequality $(\frac{1}{2}\triangle_s h)\leq m|\partial_B\bar{\partial}_B h|_s^2$. Equivalently, we have

$$\begin{aligned} & (\tfrac{\partial}{\partial s}-\tfrac{1}{4}\triangle_s)\{e^{1-s}(|dh_{d\tilde{\eta}_s}|_s+\epsilon a(s-1)\triangle_s h_{d\tilde{\eta}_s})\} \\ \leq\ & e^{1-s}\triangle_s h_{d\tilde{\eta}_s}(\epsilon a-\tfrac{1+\epsilon a(s-1)}{4m}\triangle_s h_{d\tilde{\eta}_s}). \end{aligned} \tag{7.18}$$

Then, (7.18) implies that $e^{1-s}(|dh_{d\tilde{\eta}_s}|_s+\epsilon a(s-1)\triangle_s h_{d\tilde{\eta}_s})\leq 16e^{4(m+1)}\epsilon^2$ for $s\in[1,2]$. Otherwise at the point of $[1,2]\times M$ where it fails to hold for the first time $1<t_0\leq 2$, we have $e^{1-s}\epsilon a(s-1)\triangle_s h_{d\tilde{\eta}_s}\geq 8e^{4(m+1)}\epsilon^2$ and then $\triangle_s h_{d\tilde{\eta}_s}\geq 32me^{4(m+1)}\epsilon$. But, from (7.18), we have $\triangle_s h_{d\tilde{\eta}_s}\leq\epsilon$ at the point, which is a contradiction. So, we have

$$\triangle_s h_{d\tilde{\eta}_s}\leq 64me^{4m+5}\epsilon \quad for \quad s=2, \tag{7.19}$$

and then

$$|S^T_{d\tilde{\eta}_s}-2m(m+1)|\leq 32me^{4m+5}\epsilon \quad for \quad s=2. \tag{7.20}$$

□

Corollary 7.6. *Let (M,ξ,η,Φ,g) be a compact Sasakian manifold with $[d\eta]_B=\frac{2\pi}{m+1}c_1^B(M,\mathcal{F}_\xi)$. If the $\mathcal{K}$ energy functional $\mathcal{V}_{d\eta}$ is bounded below on the space $\mathcal{H}(\xi,\eta,\Phi,g)$, then we have the Miyaoka-Yau type inequality (7.9).*

As an application of theorem 4.3 and lemma 7.2, we have the following proposition.

Proposition 7.7. *Let (M,ξ,η,Φ,g) be a compact Sasakian manifold with $[d\eta]_B=\frac{2\pi}{m+1}c_1^B(M,\mathcal{F}_\xi)$ and*

$$\int_M (2c_2^B(M,\mathcal{F}_\xi)-\frac{m}{m+1}c_1^B(M,\mathcal{F}_\xi)^2)\wedge\frac{(\frac{1}{2}d\eta)^{m-2}}{(m-2)!}\wedge\eta=0. \tag{7.21}$$

If $F_{d\eta}$ (or $\mathcal{V}_{d\eta}$) is proper in the space $\mathcal{H}(\xi,\eta,\Phi,g)$, then there must exists a Sasakian metric $(\xi,\eta',\Phi',g')\in\mathcal{S}(\xi,\bar{J})$ with constant curvature 1. Furthermore, if M is simply connected, then (M,g') is isometric to a unit sphere.

Proof. By theorem 4.3, there exists a Sasakian-Einstein $(\xi,\eta',\Phi',g')\in\mathcal{S}(\xi,\bar{J})$. By lemma 7.2, formula (7.5) and the condition (7.21), we have

$$Q_{i\bar{j}k\bar{l}}=R^T_{i\bar{j}k\bar{l}}-2(g^T_{i\bar{j}}g^T_{k\bar{l}}+g^T_{i\bar{l}}g^T_{k\bar{j}}), \tag{7.22}$$

i.e. (ξ,η',Φ',g') is of constant transverse holomorphic bisectional curvature. On the other hand, using the relation (2.5) of the transverse curvature tensor and the Riemann curvature tensor (or see proposition 7.2 in [31]), it is not hard to see that the Riemannian manifold (M,g') is of constant curvature 1. □

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